Karbala International Journal of Modern Science

Volume 5 | Issue 1 Article 7

2019

Analytical wave solutions of the space time fractional modified regularized long wave equation involving the conformable fractional derivative

M. Hafiz Uddin
Jessore University of Science and Technology, hafiz_uddin81@yahoo.com

Md. Ashrafuzzaman Khan
University of Rajshahi, akhanmath@yahoo.com

M. Ali Akbar
University of Rajshahi, ali_math74@yahoo.com

Md. Abdul Haque
University of Rajshahi, k.khanru@yahoo.com

Analytical wave solutions of the space time fractional modified regularized long wave equation involving the conformable fractional derivative

Abstract

The space time fractional modified regularized long wave equation is a model equation to the gravitational water waves in the long-wave occupancy, shallow waters waves in coastal seas, the hydro-magnetic waves in cold plasma, the phonetic waves in dissident quartz and phonetic gravitational waves in contractible liquids. In nonlinear science and engineering, the mentioned equation is applied to analyze the one way tract of long waves in seas and harbors. In this study, the closed form traveling wave solutions to the above equation are evaluated due to conformable fractional derivatives through double (G'/G,1/G)-expansion method and the Exp-function method. The existence of chain rule and the derivative of composite function permit the nonlinear fractional differential equations (NLFDEs) converted into ODEs using wave transformation. The obtain solutions are very much effective to analyze the gravitational water waves in the long-wave occupancy, shallow waters waves in coastal seas, one way tract of long waves in seas and harbors. These two methods are efficient, convenient, and computationally attractive.

Keywords

The space time fractional modified regularized long wave equation, conformable fractional derivative, traveling wave solution, double (G'/G, 1/(G))method, the Exp-function method

1. Introduction

Fractional calculus and hence the fractional order nonlinear partial differential equations have drown the great attention to many researchers for their importance to depict the inner mechanisms of the nature of real world. The fractional order differential equations are broadly used as generalizations of conventional differential equations with integral order to explain different intricate phenomena in numerous fields including the diffusion of biological populations, the signal processing, plasma physics, optical fiber, chemical kinematics, solid state physics, electric circuit, fluid flow, control theory and other areas [1–6]. The fractional order models perform various scale, namely, nanoscale, microscale, mesoscale and macroscale. The human body extremely prepared poly-layer part is especially competent model system for employing fractional calculus. Thus, the exploration of exact solutions to NLFDEs is becoming the key interest to the researchers and plays substantial role in nonlinear science. As a result, various methods have been established for searching the exact solution to NLFDEs, for example, the Adomian's decomposition method [7–9], the differential transformation method [10,11], the variational iteration method [12–14], the homotopy analysis method [15,16], the homotopy perturbation method [17,18], the finite element method [19], the (G'/G) -expansion method [20–24], the exp-function method [25–27], the fractional sub-equation method [28–30], the first integral method [31,32], the double $(G'/G, 1/G)$ -expansion method [33–35] and many more.

The suggested equation is a model equation to the gravity water waves, shallow waters waves in coastal seas, the hydro-magnetic waves in cold plasma and phonetic gravitational waves in contractible liquids. The above mentioned equation has been investigated for its exact analytic solutions through the improved Riccati expansion method [36]. Kaplan et al. [37] also has recently constructed exact solutions to the same equation by means of the MSE method. To our comprehension, the suggested equation has not been studied through the double $(G'/G, 1/G)$ -expansion method and Exp-function method. So, the goal of this study is to establish some fresh and further general exact solutions to above mentioned equation using the double $(G'/G, 1/G)$ -expansion method and Exp-function method.

The remaining part of the article is scheduled as: In section 2, we have introduced the definition and preliminaries, in section 3, the double $(G'/G, 1/G)$ -expansion method and the Exp-function method have been described. In section 4, we have established the exact solution to the suggested equation by above mentioned methods. In section 5, we have exposed the graphical representation and discussion and also in section 6, comparisons of results have been drawn. In lattermost part the conclusions are given.

2. Definition and preliminaries

Suppose $f:[0,\infty]\to\mathbb{R}$, be a function. The α -order "conformable derivative'' of f is stated as [38]:

$$T_\alpha(f)(t)=\lim_{\varepsilon\to 0}\frac{f(t+\varepsilon t^{1-\alpha})-f(t)}{\varepsilon} \tag{2.1}$$

For every $t>0, \alpha\in(0,1)$. If f is α -differentiable in some $(0,a)$, $a>0$, and $\lim_{t\to 0^+} f^{(\alpha)}(t)$ exists, then define $f^{(\alpha)}(0)=\lim_{t\to 0^+} f^{(\alpha)}(t)$. The following theorems point out few axioms that are satisfied conformable derivative.

Theorem 1. *Consider $\alpha\in[0,1]$ and let us supposed f, g be α-differentiable at a point $t>0$. Therefore*

- $T_\alpha(cf+dg)=cT_\alpha(f)+dT_\alpha(g)$, *for all* $c,d\in\mathbb{R}$
- $T_\alpha(t^p)=pt^{p-\alpha}$, *for all* $p\in\mathbb{R}$
- $T_\alpha(c)=0$, *for all constant function* $f(t)=c$.
- $T_\alpha(fg)=fT_\alpha(g)+gT_\alpha(f)$.
- $T_\alpha\left(\frac{f}{g}\right)=\frac{gT_\alpha(f)-fT_\alpha(g)}{g^2}$.
- *In addition, if f is differentiable, then* $T_\alpha(f)(t)=t^{1-\alpha}\frac{df}{dt}$.

Some more properties including the chain rule, Gronwall's inequality, some integration techniques, Laplace transform, Tailor series expansion and exponential function with respect to the conformable fractional derivative are explained in Ref. [38].

Theorem 2. *Let f be an α-differentiable function in conformable differentiable and suppose that g is also differentiable and defined in the range of f. Then*

$$T_\alpha(f\circ g)(t)=t^{1-\alpha}g'(t)f_g(t). \tag{2.2}$$

https://doi.org/10.33640/2405-609X.1010

3. Outline of the methods

In this part, we discuss the principal part of proposed methods to examine exact traveling wave solutions to the NLFDEs. Supposed the general NLFDE is as form

$$P\left(u, D_t^{\alpha}u, D_x^{\beta}u, D_t^{\alpha}D_t^{\alpha}u, D_t^{\alpha}D_x^{\beta}u, D_x^{\beta}D_x^{\beta}, \ldots \ldots \ldots\right) = 0,\ \ 0<\alpha\le 1,\ 0<\beta\le 1, \tag{3.1}$$

here u represents an unidentified function of spatial derivative x and temporal derivative $t$and Prepresents a polynomial of $u(x,t)$ and its derivatives in which the maximum order of derivatives and nonlinear terms of the maximum order are associated. Consider the wave transformation

$$\xi = k\frac{x^{\beta}}{\beta} + c\frac{t^{\alpha}}{\alpha}, u(x,t) = u(\xi), \tag{3.2}$$

where c and k are non-zero arbitrary constants.

Applying this wave transformation in (3.1), it is rewritten as:

$$R\left(u, u', u'', u''', \ldots \ldots \ldots\right) = 0, \tag{3.3}$$

here the superscripts indicate the ordinary derivative of u.

3.1. The double$(G'/G, 1/G)$-expansion method

In this subsection, the core part of the double $(G'/G, 1/G)$-expansion method to evaluate the exact traveling wave solution of the NLFDEs has been illustrated. Let us suppose the ordinary differential equation of order two

$$G''(\xi) + \lambda G(\xi) = \mu \tag{3.4}$$

and the following relations

$$\phi = G'/G, \psi = 1/G. \tag{3.5}$$

Thus, it provides

$$\phi' = -\phi^2 + \mu\psi - \lambda,\ \ \psi' = -\phi\psi. \tag{3.6}$$

The solutions to Eq. (3.4) depend on λas $\lambda<0, \lambda>0$ and $\lambda = 0$.

When $\lambda<0$, the general solution to Eq. (3.4) is

$$G(\xi) = A_1 \sinh\left(\sqrt{-\lambda}\ \xi\right) + A_2 \cosh\left(\sqrt{-\lambda}\ \xi\right) + \frac{\mu}{\lambda}, \tag{3.7}$$

In view of that, we obtain

$$\psi^2 = \frac{-\lambda}{\lambda^2\sigma + \mu^2}\left(\phi^2 - 2\mu\psi + \lambda\right), \tag{3.8}$$

where $\sigma = A_1^2 - A_2^2$.

If $\lambda>0$, the solution to (3.4) as follows:

$$G(\xi) = A_1 \sin\left(\sqrt{\lambda}\ \xi\right) + A_2 \cos\left(\sqrt{\lambda}\ \xi\right) + \frac{\mu}{\lambda}, \tag{3.9}$$

As a result, we attain

$$\psi^2 = \frac{\lambda}{\lambda^2\sigma - \mu^2}\left(\phi^2 - 2\mu\psi + \lambda\right), \tag{3.10}$$

where $\sigma = A_1^2 + A_2^2$.

When $\lambda = 0$, the solution to Eq. (3.4) is

$$G(\xi) = \frac{\mu}{2}\xi^2 + A_1\xi + A_2, \tag{3.11}$$

Therefore, we find

$$\psi^2 = \frac{1}{A_1^2 - 2\mu A_2}\left(\phi^2 - 2\mu\psi\right). \tag{3.12}$$

where $A_1$and A_2 are arbitrary constants.

Step 1: Consider the solution to (3.3) have been revealed as a polynomial inϕ and ψ of the prescribe type:

$$u(\xi) = \sum_{i=0}^{N} a_i\phi^i + \sum_{i=1}^{N} b_i\phi^{i-1}\psi, \tag{3.13}$$

Where ina_i,b_i are constants can be evaluated afterword.

Step 2: Balancing the maximum number of derivatives in linear and nonlinear terms appearing in Eq. (3.3) fixed the positive integer$N$.

Step 3: Setting (3.13) into (3.3) together with (3.6) and (3.8) it reduces to a polynomial in ϕ and ψ, where degree of ψ is single. Comparing the like terms of polynomial to zero gives an arrangement of algebraic equations that is probed by utilizing computational software yields the values of a_i, b_i, μ, A_1, A_2 and λ where $\lambda<0$, which provide hyperbolic function solutions.

Step 4: Likewise, we inspect the values of a_i, b_i, μ, A_1, A_2 and λ when $\lambda>0$ and $\lambda = 0$, yield the trigonometric and rational solutions separately.

3.2. The Exp-function method

In this subsection, the principal parts of the Exp-function technique are portrayed for searching the traveling wave solution to the NLFDEs.

Step 1: As indicated by the Exp-function technique, the arrangement is to be communicated in the shape

$$u(\xi)=\frac{\sum_{n=-c}^{d}p_n\exp(n\xi)}{\sum_{m=-p}^{q}q_m\exp(m\xi)}, \tag{3.14}$$

where $c,\ d,\ p$ and q are unknown positive integer which can be evaluated afterwards and p_n,q_m are unidentified constants.

Step 2: We balance the maximum order nonlinear term and the linear term of maximum order in (3.3) and substitute (3.14) into (3.3) evaluated c and p, and the balance of lowest order linear and nonlinear terms yield the values of d and q.

Step 3: Substitute (3.14) into (3.3) and compare $\exp(n\xi)$ to zero, acquire an arrangement of set of equations for p_n, q_m, c and k. Then, unravel the set along with the guide of computer software like Maple to decide the constants.

Step 4: Substituting the values showed up in Step 3 into (3.14), we get exact solutions to the NLFDEs in (3.1).

4. Formulation of the solutions

In this part, we search further comprehensive analytic wave solutions for desired equation by means of above suggested methods. Let us assume that the regularized long wave equation is as follows:

$$D_t^{\alpha}u+vD_x^{\alpha}u+\eta u^2D_x^{\alpha}u-\tau D_t^{\alpha}D_x^{2\alpha}u=0,\ \ 0<\alpha\le 1, \tag{4.1}$$

where$\tau,\ \eta,\ v$ are arbitrary constants. This equation was first inferred by Benjamin et al. [39] in 1972 to assign estimation for surface long waves in oceans and harbor. It is regarded as the variant to the KdV equation which exhibited to oversee a substantial number, for example, shallow waters and plasma waves in beach front oceans. It is competent to search the scientific model of gravitational water waves in the long-wave occupancy, the phonetic waves in dissident quartz, the hydro-magnetic waves in chill plasma and phonetic gravitational waves in contractible liquids. For Eq. (4.1), we recommend the subsequent transformation

$$\xi=x+\omega\frac{t^{\alpha}}{\alpha},u(x,t)=u(\xi), \tag{4.2}$$

where ω be the velocity of traveling wave. Utilizing Eq. (4.2), into Eq. (4.1) reduces to the next integral order ordinary differential equation (ODE):

$$(\omega+v)u'+\eta u^2u'-\omega\tau u'''=0. \tag{4.3}$$

Integrating (4.3) with zero constant, we attain

$$(\omega+v)u+\frac{\eta u^3}{3}-\tau\omega u''=0. \tag{4.4}$$

4.1. Solutions due to the double $(G'/G,1/G)$-expansion method

Assuming homogeneous equivalence to the maximum order derivative and the maximum order nonlinear term showing up in Eq. (4.4), the arrangement Eq. (3.13) takes the shape:

$$u(\xi)=a_0+a_1\phi(\xi)+b_1\psi(\xi), \tag{4.5}$$

here a_0,a_1,b_1 are constants to be resolved.

Case 1. For $\lambda<0$, embedding Eq. (4.5) into (4.4) alongside with Eqs. (3.6) and (3.8) yields an arrangement of mathematical equations and explaining these arithmetical equations by utilizing computer algebra like Maple, we accomplish the subsequent outcomes:

$$a_0=0,\ a_1=\pm\sqrt{\frac{3v\tau}{\eta(\tau\lambda-2)}},$$

$$b_1=\pm\sqrt{\frac{-3v\tau\left(\mu^2+\lambda^2\sigma\right)}{\eta\lambda(\tau\lambda-2)}},\text{and }\omega=\frac{2v}{\tau\lambda-2}.$$

Inserting the on top of values into (4.5), we find the solution to Eq. (4.1) in the form:

$$u_{1_1}(x,t)=\pm\sqrt{\frac{3v\tau}{\eta(\tau\lambda-2)}}\times\frac{\sqrt{-\lambda}\left(A_1\cosh\left(\sqrt{-\lambda}\left(x+\frac{2v}{\tau\lambda-2}\frac{t^{\alpha}}{\alpha}\right)\right)+A_2\sinh\left(\sqrt{-\lambda}\left(x+\frac{2v}{\tau\lambda-2}\frac{t^{\alpha}}{\alpha}\right)\right)\right)}{A_1\sinh\left(\sqrt{-\lambda}\left(x+\frac{2v}{\tau\lambda-2}\frac{t^{\alpha}}{\alpha}\right)\right)+A_2\cosh\left(\sqrt{-\lambda}\left(x+\frac{2v}{\tau\lambda-2}\frac{t^{\alpha}}{\alpha}\right)\right)+\frac{\mu}{\lambda}}\pm\sqrt{\frac{-3v\tau\left(\mu^2+\lambda^2\sigma\right)}{\eta\lambda(\tau\lambda-2)}}$$
$$\times\frac{1}{A_1\sinh\left(\sqrt{-\lambda}\left(x+\frac{2v}{\tau\lambda-2}\frac{t^{\alpha}}{\alpha}\right)\right)+A_2\cosh\left(\sqrt{-\lambda}\left(x+\frac{2v}{\tau\lambda-2}\frac{t^{\alpha}}{\alpha}\right)\right)+\frac{\mu}{\lambda}} \tag{4.6}$$

where $\sigma=A_1^2-A_2^2$.Since A_1 and A_2 are basic constants, one might chose self-assertively theirs values.

Hence by setting $A_1 = 0$ $A_2 \neq 0$(or $A_1 \neq 0$, $A_2 = 0$) and $\mu = 0$ in (4.6), we find the solitary wave solution

$$u_{1_2}(x,t) = \pm\sqrt{\frac{3\nu\tau}{\eta(\tau\lambda-2)}} \times \sqrt{-\lambda}\tanh\left(\sqrt{-\lambda}\left(x+\frac{2\nu}{\tau\lambda-2}\frac{t^\alpha}{\alpha}\right)\right)$$

$$\pm\sqrt{\frac{-3\nu\tau\lambda\sigma}{\eta(\tau\lambda-2)}} \times sech\left(\sqrt{-\lambda}\left(x+\frac{2\nu}{\tau\lambda-2}\frac{t^\alpha}{\alpha}\right)\right). \tag{4.7}$$

$$u_{1_3}(x,t) = \pm\sqrt{\frac{3\nu\tau}{\eta(\tau\lambda-2)}} \times \sqrt{-\lambda}\coth\left(\sqrt{-\lambda}\left(x+\frac{2\nu}{\tau\lambda-2}\frac{t^\alpha}{\alpha}\right)\right)$$

$$\pm\sqrt{\frac{-3\nu\tau\lambda\sigma}{\eta(\tau\lambda-2)}} \times cosech\left(\sqrt{-\lambda}\left(x+\frac{2\nu}{\tau\lambda-2}\frac{t^\alpha}{\alpha}\right)\right). \tag{4.8}$$

Case 2. In a comparative way, when $\lambda > 0$, substituting Eq. (4.5) into (4.4) together with (3.6) and (3.10) yield an arrangement of mathematical equations for a_0,a_1, b_1, c and k and solving these equations, we get the following results:

$$a_0 = 0,\ a_1 = \pm\sqrt{\frac{3\nu\tau}{\eta(\tau\lambda-2)}},$$

$$b_1 = \pm\sqrt{\frac{3\nu\tau(\lambda^2\sigma-\mu^2)}{\eta\lambda(\tau\lambda-2)}},\ \text{and}\ \omega = \frac{2\nu}{\tau\lambda-2}.$$

The substitution of these outcomes in Eq. (4.5) possesses the following expression for the general solution of Eq. (4.1):

$$u_{1_4}(x,t) = \pm\sqrt{\frac{3\nu\tau}{\eta(\tau\lambda-2)}} \times \frac{\sqrt{\lambda}\left(A_1\cos\left(\sqrt{\lambda}\left(x+\frac{2\nu}{\tau\lambda-2}\frac{t^\alpha}{\alpha}\right)\right) - A_2\sin\left(\sqrt{\lambda}\left(x+\frac{2\nu}{\tau\lambda-2}\frac{t^\alpha}{\alpha}\right)\right)\right)}{A_1\sin\left(\sqrt{\lambda}\left(x+\frac{2\nu}{\tau\lambda-2}\frac{t^\alpha}{\alpha}\right)\right) + A_2\cos\left(\sqrt{\lambda}\left(x+\frac{2\nu}{\tau\lambda-2}\frac{t^\alpha}{\alpha}\right)\right) + \frac{\mu}{\lambda}} \pm\sqrt{\frac{3\nu\tau(\lambda^2\sigma-\mu^2)}{\eta\lambda(\tau\lambda-2)}}$$
$$\times\frac{1}{A_1\sin\left(\sqrt{\lambda}\left(x+\frac{2\nu}{\tau\lambda-2}\frac{t^\alpha}{\alpha}\right)\right) + A_2\cos\left(\sqrt{\lambda}\left(x+\frac{2\nu}{\tau\lambda-2}\frac{t^\alpha}{\alpha}\right)\right) + \frac{\mu}{\lambda}} \tag{4.9}$$

where $\sigma = A_1^2 + A_2^2$.

If the unknown parameters are assigned as $A_1 = 0$ $A_2 \neq 0$ (or $A_1 \neq 0$, $A_2 = 0$) and $\mu = 0$ in Eq. (4.9), we acquire the solitary wave solution

$$u_{1_5}(x,t) = \pm\sqrt{\frac{3\nu\tau}{\eta(\tau\lambda-2)}} \times \sqrt{\lambda}\tan\left(\sqrt{\lambda}\left(x+\frac{2\nu}{\tau\lambda-2}\frac{t^\alpha}{\alpha}\right)\right) \pm\sqrt{\frac{3\nu\tau\lambda\sigma}{\eta(\tau\lambda-2)}} \times sec\left(\sqrt{\lambda}\left(x+\frac{2\nu}{\tau\lambda-2}\frac{t^\alpha}{\alpha}\right)\right). \tag{4.10}$$

$$u_{1_6}(x,t) = \pm\sqrt{\frac{3\nu\tau}{\eta(\tau\lambda-2)}} \times \sqrt{\lambda}\cot\left(\sqrt{\lambda}\left(x+\frac{2\nu}{\tau\lambda-2}\frac{t^\alpha}{\alpha}\right)\right) \pm\sqrt{\frac{3\nu\tau\lambda\sigma}{\eta(\tau\lambda-2)}} \times cosec\left(\sqrt{\lambda}\left(x+\frac{2\nu}{\tau\lambda-2}\frac{t^\alpha}{\alpha}\right)\right). \tag{4.11}$$

Case 3. In the alike approach when $\lambda = 0$, using Eqs. (4.5) and (4.4) along with(3.6) and (3.12), we will achieve a set of mathematical equations, whose solutions are

$$a_0 = 0,\ a_1 = \pm\sqrt{\frac{3\nu\tau}{2\eta}},\ b_1 = \pm\sqrt{\frac{6\mu A_2\tau\nu - 3\tau A_1^2\nu}{2\eta}},\ \text{and}\ \omega = -\nu.$$

Making use of these values into Eq. (4.5), produces the solution to Eq. (4.1) as

$$u_{1_7}(x,t)=\pm\sqrt{\frac{3v\tau}{2\eta}}\times\frac{\mu\left(x-v\frac{t^\alpha}{\alpha}\right)+A_1}{\frac{\mu}{2}\left(x-v\frac{t^\alpha}{\alpha}\right)^2+A_1\left(x-v\frac{t^\alpha}{\alpha}\right)+A_2}\pm\sqrt{\frac{6\mu A_2\tau v-3\tau A_1^2 v}{2\eta}}\times\frac{1}{\frac{\mu}{2}\left(x-v\frac{t^\alpha}{\alpha}\right)^2+A_1\left(x-v\frac{t^\alpha}{\alpha}\right)+A_2}. \quad (4.12)$$

It is superb to see that the voyaging wave solutions u_{1_1}- $u_{1_7}$of our desired equation are extensively new and general. These gained solutions have not been recorded in the former study. These solutions are convenient to designate the gravitational water waves in the long-wave occupancy, the phonetic waves in dissident quartz and phonetic gravitational waves in contractible liquid.

4.2. Solution by the Exp-function method

Considering the homogeneous equivalence the solution of Eq. (3.14) takes the form:

$$u(\xi)=\frac{p_1\exp(\xi)+p_0+p_{-1}\exp(-\xi)}{q_1\exp(\xi)+q_0+q_{-1}\exp(-\xi)}. \quad (4.13)$$

Substituting Eq. (4.13) into (4.4), leads a equation in $\exp(n\xi)$, where n is any whole number. Inserting each coefficient of this equation to zero, yields a cluster of mathematical equations (For straightforwardness we discarded here) for p_i's, q_i's and ω.Solving this mathematical equations by computer algebra like Maple provides the results:

Set 1: $\omega=\frac{v}{\tau-1}$, $p_{-1}=0$, $p_0=p_0$, $p_1=0$, $q_{-1}=q_{-1}$, $q_0=0$ and $q_1=-\frac{1}{24}\frac{\eta p_0^2(\tau-1)}{q_{-1}v\tau}$.

Set 2: $\omega=-\frac{2v}{\tau+2}$, $p_{-1}=0$, $p_0=-q_0\sqrt{-\frac{3v\tau}{\eta(\tau+2)}}$, $p_1=q_1\sqrt{-\frac{3v\tau}{\eta(\tau+2)}}$, $q_{-1}=0$, $q_0=q_0$ and $q_1=q_1$.

Set 3: $\omega=-\frac{1}{3}\frac{3q_0^2v+\eta p_0^2}{q_0^2}$, $q_{-1}=\frac{p_0q_{-1}}{q_0}$, $p_0=p_0$, $p_1=0$, $q_{-1}=q_{-1}$, $q_0=q_0$ and $q_1=0$.

Set 4: $\omega=-\frac{2v}{\tau+2}$, $p_{-1}=-q_{-1}\sqrt{-\frac{3v\tau}{\eta(\tau+2)}}$, $p_0=q_0\sqrt{-\frac{3v\tau}{\eta(\tau+2)}}$, $p_1=0$, $q_{-1}=q_{-1}$, $q_0=q_0$ and $q_1=0$.

Set 5: $\omega=-\frac{v}{2\tau+2}$, $p_{-1}=-q_{-1}\sqrt{-\frac{6v\tau}{\eta(2\tau+1)}}$, $p_0=0$, $p_1=q_1\sqrt{-\frac{6v\tau}{\eta(2\tau+1)}}$, $q_{-1}=q_{-1}$, $q_0=0$ and $q_1=q_1$.

Set 6: $\omega=-\frac{1}{3}\frac{\eta p_1^2+3vq_1^2}{q_1^2}$, $p_{-1}=\frac{p_1q_{-1}}{q_1}$, $p_0=\frac{p_1q_0}{q_1}$, $p_1=p_1$, $q_{-1}=q_{-1}$, $q_0=q_0$ and $q_1=q_1$.

Set 7: $\omega=-\frac{2v}{\tau+2}$, $p_{-1}=-\frac{q_0^2}{4q_1}\sqrt{-\frac{3v\tau}{\eta(\tau+2)}}$, $p_0=0$, $p_1=q_1\sqrt{-\frac{3v\tau}{\eta(\tau+2)}}$, $q_{-1}=\frac{q_0^2}{4q_1}$, $q_0=q_0$ and $q_1=q_1$.

Set 8: $\omega=-\frac{2v}{\tau+2}$, $p_{-1}=\frac{1}{4}\frac{3q_0^2v\tau+\eta p_0^2\tau+2\eta p_0^2}{q_1}$, $p_0=p_0$, $p_1=p_1$, $q_{-1}=\frac{1}{12}\frac{3q_0^2v\tau+\eta p_0^2\tau+2\eta p_0^2}{p_1\tau v\sqrt{\frac{-\eta(\tau+2)}{3v\tau}}}$, $q_0=q_0$ and $q_1=p_1\sqrt{\frac{-\eta(\tau+2)}{3v\tau}}$.

Set 9: $\omega=\frac{1}{3}\frac{vq_{-1}^2+\eta p_{-1}^2}{q_{-1}^2}$, $p_{-1}=p_{-1}$, $p_0=0$, $p_1=0$, $q_{-1}=q_{-1}$, $q_0=0$ and $q_1=0$.

In perspective of the above outcomes, we acquire the following generalized solitary wave solutions for $\eta=-1$.

$$u_{2_1}(x,t)=-\frac{p_0}{\frac{1}{24}\frac{\eta p_0^2(\tau-1)}{q_{-1}v\tau}\exp\left[\left(x+\frac{v}{\tau-1}\frac{t^\alpha}{\alpha}\right)\right]+q_{-1}\exp\left[-\left(x+\frac{v}{\tau-1}\frac{t^\alpha}{\alpha}\right)\right]}. \quad (4.14)$$

$$u_{2_2}(x.t)=\frac{q_1\sqrt{\frac{3v\tau}{(\tau+2)}}\exp\left[\left(x-\frac{2v}{\tau+2}\frac{t^\alpha}{\alpha}\right)\right]-q_0\sqrt{\frac{3v\tau}{(\tau+2)}}}{q_1\exp\left[\left(x-\frac{2v}{\tau+2}\frac{t^\alpha}{\alpha}\right)\right]+q_0}. \quad (4.15)$$

$$u_{2_3}(x,t)=\frac{p_0+\frac{p_0q_{-1}}{q_0}\exp\left[-\left(x-\frac{1}{3}\frac{3q_0^2v-p_0^2}{q_0^2}\frac{t^\alpha}{\alpha}\right)\right]}{q_0+q_{-1}\exp\left[-\left(x-\frac{1}{3}\frac{3q_0^2v-p_0^2}{q_0^2}\frac{t^\alpha}{\alpha}\right)\right]}. \quad (4.16)$$

$$u_{2_4}(x,t)=\frac{q_0\sqrt{\frac{3v\tau}{(\tau+2)}}-q_{-1}\sqrt{\frac{3v\tau}{(\tau+2)}}\exp\left[-\left(x-\frac{2v}{\tau+2}\frac{t^\alpha}{\alpha}\right)\right]}{q_0+q_{-1}\exp\left[-\left(x-\frac{2v}{\tau+2}\frac{t^\alpha}{\alpha}\right)\right]}. \tag{4.17}$$

$$u_{2_5}(x,t)=\frac{q_1\sqrt{\frac{6v\tau}{(2\tau+1)}}\exp\left[\left(x-\frac{v}{2\tau+1}\frac{t^\alpha}{\alpha}\right)\right]-q_{-1}\sqrt{\frac{6v\tau}{(2\tau+1)}}\exp\left[-\left(x-\frac{v}{2\tau+1}\frac{t^\alpha}{\alpha}\right)\right]}{q_1\exp\left[\left(x-\frac{v}{2\tau+1}\frac{t^\alpha}{\alpha}\right)\right]+q_{-1}\exp\left[-\left(x-\frac{v}{2\tau+1}\frac{t^\alpha}{\alpha}\right)\right]}. \tag{4.18}$$

$$u_{2_6}(x,t)=\frac{p_1\exp\left[\left(x+\frac{1}{3}\frac{p_1^2+3vq_1^2}{q_1^2}\frac{t^\alpha}{\alpha}\right)\right]+\frac{p_1q_0}{q_1}+\frac{p_1q_{-1}}{q_1}\exp\left[-\left(x+\frac{1}{3}\frac{p_1^2+3vq_1^2}{b_1^2}\frac{t^\alpha}{\alpha}\right)\right]}{q_1\exp\left[\left(x+\frac{1}{3}\frac{p_1^2+3vq_1^2}{q_1^2}\frac{t^\alpha}{\alpha}\right)\right]+q_0+q_{-1}\exp\left[-\left(x+\frac{1}{3}\frac{p_1^2+3vq_1^2}{q_1^2}\frac{t^\alpha}{\alpha}\right)\right]}. \tag{4.19}$$

$$u_{2_7}(x,t)=\frac{q_1\sqrt{\frac{3v\tau}{(\tau+2)}}\exp\left[\left(x-\frac{2v}{\tau+2}\frac{t^\alpha}{\alpha}\right)\right]-\frac{q_0^2}{4q_1}\sqrt{\frac{3v\tau}{(\tau+2)}}\exp\left[-\left(x-\frac{2v}{\tau+2}\frac{t^\alpha}{\alpha}\right)\right]}{q_1\exp\left[\left(x-\frac{2v}{\tau+2}\frac{t^\alpha}{\alpha}\right)\right]+q_0+\frac{q_0^2}{4q_1}\exp\left[-\left(x-\frac{2v}{\tau+2}\frac{t^\alpha}{\alpha}\right)\right]}. \tag{4.20}$$

$$u_{2_8}(x,t)=\frac{p_1\exp\left[\left(x-\frac{2v}{\tau+2}\frac{t^\alpha}{\alpha}\right)\right]+p_0+\frac{1}{4}\frac{3q_0^2v\tau-p_0^2\tau-2p_0^2}{q_1}\exp\left[-\left(x-\frac{2v}{\tau+2}\frac{t^\alpha}{\alpha}\right)\right]}{p_1\sqrt{\frac{(\tau+2)}{3v\tau}}\exp\left[\left(x-\frac{2v}{\tau+2}\frac{t^\alpha}{\alpha}\right)\right]+q_0+\frac{1}{12}\frac{3q_0^2v\tau-p_0^2\tau-2p_0^2}{p_1\tau v\sqrt{\frac{(\tau+2)}{3v\tau}}}\exp\left[-\left(x-\frac{2v}{\tau+2}\frac{t^\alpha}{\alpha}\right)\right]}. \tag{4.21}$$

$$u_{2_9}(x,t)=\frac{p_{-1}\exp\left[-\left(x-\frac{1}{3}\frac{vq_{-1}^2+\eta p_{-1}^2}{q_{-1}^2}\frac{t^\alpha}{\alpha}\right)\right]}{q_{-1}\exp\left[-\left(x-\frac{1}{3}\frac{vq_{-1}^2+\eta p_{-1}^2}{q_{-1}^2}\frac{t^\alpha}{\alpha}\right)\right]} \tag{4.22}$$

In particular case if $q_1=q_{-1}$, Eq. (4.18) is simplified and offers the kink type solution of the form:

$$u_{2_{10}}(x,t)=\sqrt{\frac{6v\tau}{(2\tau+1)}}\tanh\left(x-\frac{v}{2\tau+1}\frac{t^\alpha}{\alpha}\right). \tag{4.23}$$

The choice $q_1=-q_{-1}$ in Eq. (4.18) gives another kink type solution.

$$u_{2_{11}}(x,t)=\sqrt{\frac{6v\tau}{(2\tau+1)}}\coth\left(x-\frac{v}{2\tau+1}\frac{t^\alpha}{\alpha}\right). \tag{4.24}$$

Putting $p_1=q_1=q_{-1}$ and $b_0=0$ in Eq. (4.19) produces the kink type solution.

$$u_{2_{12}}(x,t)=\tanh\left(x+\frac{1}{3}\frac{p_1^2+3vq_1^2}{q_1^2}\frac{t^\alpha}{\alpha}\right). \tag{4.25}$$

By setting $p_1=q_1=-q_{-1}$ and $b_0=0$ in Eq. (4.19) gives another kink type solution.

$$u_{2_{13}}(x,t)=\coth\left(x+\frac{1}{3}\frac{p_1^2+3vq_1^2}{q_1^2}\frac{t^\alpha}{\alpha}\right). \tag{4.26}$$

Inserting $q_0=2q_1=2q_{-1}$ and $q_0=2q_1=-2q_{-1}$ in Eq. (4.20) yields the solutions.

$$u_{2_{14}}(x,t)=\sqrt{\frac{3v\tau}{(\tau+2)}}\frac{1}{\coth\left(x-\frac{2v}{\tau+2}\frac{t^\alpha}{\alpha}\right)+\operatorname{cosech}\left(x-\frac{2v}{\tau+2}\frac{t^\alpha}{\alpha}\right)}. \tag{4.27}$$

$$u_{2_{15}}(x,t)=\sqrt{\frac{3v\tau}{(\tau+2)}}\frac{1}{\tanh\left(x-\frac{2v}{\tau+2}\frac{t^\alpha}{\alpha}\right)+\operatorname{sech}\left(x-\frac{2v}{\tau+2}\frac{t^\alpha}{\alpha}\right)}. \tag{4.28}$$

It is notable to comprehend that the voyaging wave solutions (4.22)−(4.28) of the suggested equation are new and especially essential on the grounds that these solutions were not founded in the earlier investigations. This diffusion equation is significant in various physical phenomena. It is inferred to assign a guess for the gravitational water waves in the long-wave occupancy, the phonetic waves in dissident quartz and the hydromagnetic waves in cool plasma.

5. Graphical representation and discussion

In this section, the graphical representation and discussion to the obtained solutions of NLFDE over suggested equations are depicted. Solutions $u_{1_2}(x, t)$, $u_{2_{10}}(x, t)$,$u_{2_{12}}(x,t)$ and $u_{2_{14}}(x,t)$ represents kink type. Kink waves are one kind of traveling waves which rise from one asymptotic state to another. Fig. 1 represent the nature of the kink type solution of $u_{1_2}(x, t)$. The nature of the shape of solution $u_{1_2}(x,t)$ is analogous to the figure of solution $u_{2_{10}}(x, t)$, $u_{2_{12}}(x, t)$and $u_{2_{14}}(x, t)$therefore for simplicity the nature of these solutions are excluded here. The solutions $u_{1_6}(x,t)$, $u_{2_{11}}(x,t)$and $u_{2_{15}}(x,t)$ obtained in this study are the single solitons solutions. Fig. 2 shows the nature of the exact singular solitons solution of $u_{1_6}(x,t)$ of the general time fractional regularized long wave equation. The shape of solutions$u_{2_{11}}(x,t)$ and $u_{2_{15}}(x,t)$ is analogous to the shape of solution$u_{1_6}(x,t)$, consequently for convenience these solutions are omitted here. Solutions$u_{1_3}(x,t)u_{1_5}(x,t)$ and $u_{2_{13}}(x,t)$ denote the exact periodic traveling wave solutions. Periodic solutions are traveling wave solutions which are periodic. Fig. 3 indicate the nature of the spike solution of $u_{1_3}(x,t)$. The figure of solutions$u_{1_5}(x, t)$and $u_{2_{13}}(x,t)$ is eliminated here for minimalism. In the end, the solutions $u_{1_7}(x, t)$represents the singular kink type. Fig. 4 denotes the exact singular kink type solution of $u_{1_7}(x,t)$.

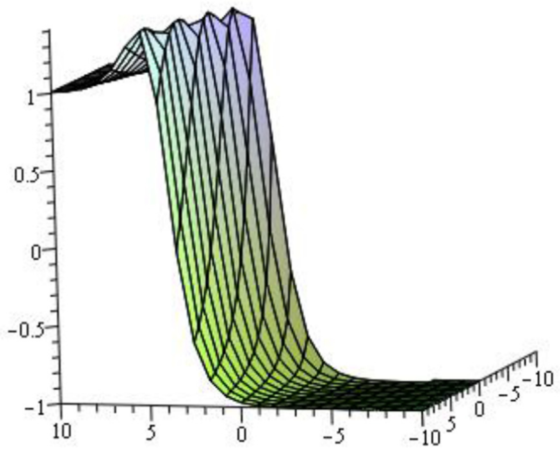

Fig. 1. Sketch of kink wave of $u_{1_2}(x,t)$ if $\lambda = -1$, $\nu = 1$, $\tau = 1$, $\eta = -1$, $\sigma = 1$, $\alpha = 1/2$ and. $-10 \le x,t \le 10$

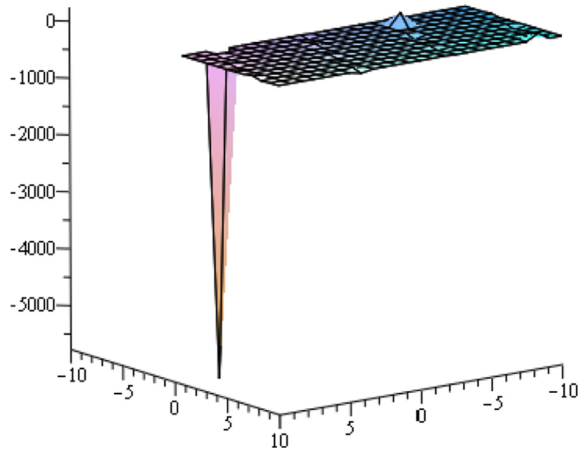

Fig. 2. Sketch of single soliton wave of $u_{1_6}(x,t)$ if $\lambda = 1$, $\nu = 1$, $\tau = -1$, $\eta = 1$, $\sigma = 1$, $\alpha = 1/2$ and $-10 \le x,t \le 10$.

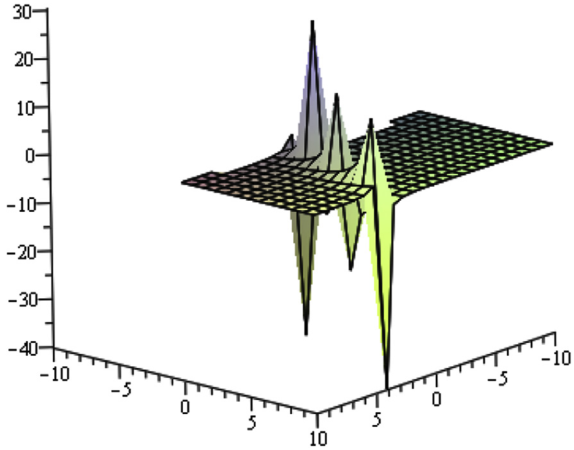

Fig. 3. Shape of spike wave of $u_{1_3}(x,t)$ if $\lambda = -1$, $\nu = 1$, $\tau = 1$, $\eta = -1$, $\sigma = 1$, $\alpha = 1/2$ and $-10 \le x,t \le 10$.

6. Comparison of results

It is remarkable to observe that some of the obtained solutions demonstrate good similarity with earlier established solutions. A comparison between Abdel-Salam and Gumma [36] solutions and our obtained solutions is presented in Tables 1 and 2.

The hyperbolic function solutions referred to the above Table is similar and if we set definite values of the arbitrary constants they are identical. It is substantial to understand that the traveling wave

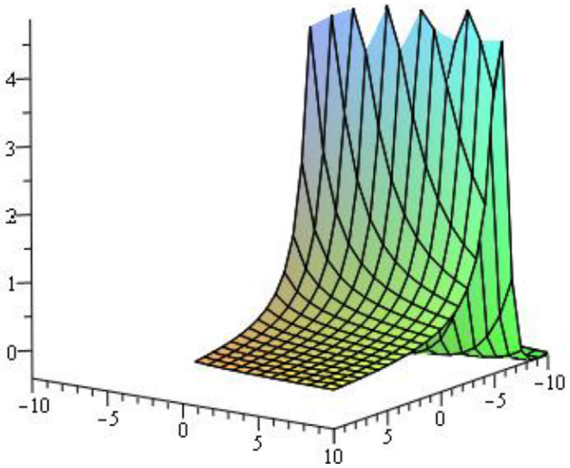

Fig. 4. Shape of singular kink type wave of $u_{1_7}(x,t)$ if $\mu = 2, A_1 = -1, A_2 = 1, v = -1, \tau = -1, \eta = 1, \alpha = 1/2$ and $-10 \le x, t \le 10$.

solution$u_{1_5}(x,t)$, $u_{1_6}(x,t)$, $u_{1_7}(x,t) u_{2_{12}}(x,t)$, $u_{2_{13}}(x,t)$, $u_{2_{14}}(x,t)$ and $u_{15}(x,t)$of the fractional modified regularized long wave equation are all new and very much important which were not originate in the previous work. These solutions are significant in various physical phenomena.

It is interesting to observe that if $\alpha = 1$ to our obtained solution the conformable fractional derivatives go into the ordinary derivative and the results coincides to some known solution of modified regularized long wave equation. In Ref. [36] the solution$u_{1_{mRLW}}$ in Eq. (37) and our achieve solutions$u_{1_2}(x,t)$, $u_{2_{10}}(x,t)$, $u_{2_{12}}(x,t)$are identical if we set definite values of arbitrary constants (Appendix A). Once more, the solution $u_{2_{mRLW}}$of [36] in Eq. (38) and our obtained solutions $u_{1_3}(x,t)$, $u_{2_{11}}(x,t)$, $u_{2_{13}}(x,t)$ are coincides if we select definite values of arbitrary constants (Appendix B).

7. Conclusion

In this study, we have obtained some fresh and further general traveling wave solutions to the NLFDE, the fractional modified regularized long wave equation. The achieved solutions and their advantages are listed below:

- The achieved solutions for this equation are competent to examine the scientific model of gravity water waves in shallow water.
- It is capable to investigate plasma waves in seaside oceans and break down the unidirectional spread of long waves in oceans and harbor.
- Additionally, the solutions is capable to clarify diverse physical incident, for example, the hydromagnetic waves in cold plasma, the phonetic waves in dissident quartz and phonetic gravitational waves in contractible liquid.

The strength of the double $(G'/G, 1/G)$-expansion and the Exp-function methods are predictable,

Table 1
Comparison between Abdel-Salam and Gumma solution and our solution by double $(G'/G, 1/G)$-expansion method.

Abdel-Salam and Gumma solution [36]	Solutions obtained in this article
If $B^2 - 4AC > 0$ and $BC \neq 0$ the solution in Eq. (34) becomes:	If $\sigma = 0$ in solution (4.7) then our solution $u_{1_2}(x,t)$ becomes:
$U_1(\xi) = \pm\sqrt{-\frac{3v\tau(B^2-4AC)}{\mu(2+B^2\tau-4AC\tau)}}\tanh(0.5\sqrt{B^2-4AC}\xi, \alpha)$	$u_{1_2}(x,t) = \pm\sqrt{\frac{3v\tau}{\eta(\tau\lambda-2)}} \times \sqrt{-\lambda}\tanh\left(\sqrt{-\lambda}\left(x+\frac{2v}{\tau\lambda-2}\frac{t^\alpha}{\alpha}\right)\right)$
If $B^2 - 4AC > 0$ and $BC \neq 0$ the solution in Eq. (35) becomes:	If $\sigma = 0$ in the solution (4.8), then the obtained solution $u_{1_3}(x,t)$ becomes:
$U_2(\xi) = \pm\sqrt{-\frac{3v\tau(B^2-4AC)}{\mu(2+B^2\tau-4AC\tau)}}\coth(0.5\sqrt{B^2-4AC}\xi, \alpha)$	$u_{1_3}(x,t) = \pm\sqrt{\frac{3v\tau}{\eta(\tau\lambda-2)}} \times \sqrt{-\lambda}\coth\left(\sqrt{-\lambda}\left(x+\frac{2v}{\tau\lambda-2}\frac{t^\alpha}{\alpha}\right)\right)$

Table 2
Comparison between Abdel-Salam and Gumma solution and our solution by Exp-function method.

Abdel-Salam and Gumma [36]	Obtained solutions
If $B^2 - 4AC > 0$ and $BC \neq 0$, the solution in Eq. (34) becomes:	If we put $q_1 = q_{-1}$ in solution (4.18) then our solution $u_{2_{10}}(x,t)$ becomes:
$U_1(\xi) = \pm\sqrt{-\frac{3v\tau(B^2-4AC)}{\mu(2+B^2\tau-4AC\tau)}}\tanh(0.5\sqrt{B^2-4AC}\xi, \alpha)$	$u_{2_{10}}(x,t) = \sqrt{\frac{6v\tau}{(2\tau+1)}}\tanh\left(x-\frac{v}{2\tau+1}\frac{t^\alpha}{\alpha}\right)$
If $B^2 - 4AC > 0$ and $BC \neq 0$, the solution in Eq. (35) becomes:	If we set $q_1 = -q_{-1}$ in solution (4.18) then our solution $u_{2_{11}}(x,t)$ becomes:
$U_2(\xi) = \pm\sqrt{-\frac{3v\tau(B^2-4AC)}{\mu(2+B^2\tau-4AC\tau)}}\coth(0.5\sqrt{B^2-4AC}\xi, \alpha)$	$u_{2_{11}}(x,t) = \sqrt{\frac{6v\tau}{(2\tau+1)}}\coth\left(x-\frac{v}{2\tau+1}\frac{t^\alpha}{\alpha}\right)$

dependable and computationally appealing. We guarantee that the recommended methods may assume huge job in additionally inquire about.

Acknowledgement

The authors would like to express their sincere thanks to the anonymous referees for their valuable comments and suggestions to improve the article. The authors also would like to convey their appreciation to Dr. Md. Mominul Islam, Professor, Department of English, University of Rajshahi, Bangladesh for his assistance in editing the English language.

Appendix A.

$$U_{1_{mRLW}} = \pm\sqrt{-\frac{3\nu\tau(B^2-4AC)}{\mu(2+B^2\tau-4AC\tau)}}\tanh\left(0.5\sqrt{B^2-4AC}\xi\right).$$

Appendix B.

$$U_{2_{mRLW}} = \pm\sqrt{-\frac{3\nu\tau(B^2-4AC)}{\mu(2+B^2\tau-4AC\tau)}}coth\left(0.5\sqrt{B^2-4AC}\xi\right).$$

Conflicts of interest

We declare that none of the authors have any competing of interests in this manuscript.

Author's contribution

Author ‘a’ wrote the manuscript and performs the analytical calculations with support from ‘b’. Both the author's contributed to the design and implementation of the research, to the analysis of results and edited the manuscript.

References

[1] K.S. Miller, B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations, Wiley, New York, 1993.
[2] I. Podlubny, Fractional Differential Equations, Academic, San Diego, CA, 1999.
[3] F.S. Mozaffari, H. Hassanabadi, H. Sobhani, W.S. Chung, On the conformable fractional quantum mechanics, J. Kor. Phys. Soc. (2017), https://doi.org/10.3938/jkps.70.348.
[4] W.S. Chung, S. Zare, H. Hassanabadi, Investigation of conformable fractional Schrodinger equation in presence of killingbeck and hyperbolic potentials, Commun. Theor. Phys. 67 (3) (2017) 250−254, https://doi.org/10.1088/0253-6102/67/3/250.
[5] W.S. Chung, S. Zarrinkamar, S. Zare, H. Hassanabadi, Scattering study of a modified cusp potential in conformable fractional formalism, J. Kor. Phys. Soc. 70 (4) (2017) 348−352, https://doi.org/10.3938/jkps.70.348.
[6] Mozaffari FS, Hassanabadi H, Sobhani H, Chung WS: Investigation of the Dirac equation by using the conformable fractional derivative. J. Kor. Phys. Soc. DOI:10.3938/jkps.72.987.
[7] A.M.A. El-Sayed, S.H. Behiry, W.E. Raslan, Adomian's decomposition method for solving an intermediate fractional advection-dispersion equation, Comput. Math. Appl. 59 (5) (2010) 1759−1765.
[8] S.S. Ray, A new approach for the application of Adomian's decomposition method for the solution to fractional space diffusion equation with insulated ends, Appl. Math. Comput. 202 (2) (2008) 544−549.
[9] A.M.A. El-Sayedand, M. Gaber, The Adomian's decomposition method for solving partial differential equation of fractional order in finite domains, Phys. Lett. 359 (3) (2006) 175−182.
[10] Z. Odibat, S. Momani, Generalized differential transform method for linear partian differential equations of fractional order, Appl. Math. Lett. 21 (2) (2008) 194−199.
[11] V.S. Erturk, Odibat Z. MomaniS, Application of generalized transformation method to multi-order fractional differential equations, Commun. Nonlinear Sci. Numer. Simul. 13 (8) (2008) 1642−1654.
[12] J. Ji, J.B. Zhang, Y.J. Dong, The fractional variational iteration method improved with the Adomian series, Appl. Math. Lett. 25 (2012) 2223−2226.
[13] A.R. Seadawy, Approximation solutions to derivative nonlinear Schrodinger equation with computational applications by variational method, Eur. Phys. J. Plus 130 (2015) 182−188.
[14] S. Guo, L. Mei, The fractional variational iteration method using He's polynomial, Phys. Lett. 375 (3) (2011) 309−313.
[15] L.N. Song, H.Q. Zhang, Solving the fractional BBM-Burger equation using the Homotopy analysis method, Chaos, Solit. Fractals 40 (4) (2009) 1616−1622.
[16] A.A.M. Arafa, S.Z. Rida, H. Mohamed, Homotopy analysis method for solving biological population model, Commun. Theor. Phys. 56 (5) (2011) 797−800.
[17] K.A. Geprel, The Homotopy perturbation method applied to the nonlinear fractional Kolmogorov-Petrovskii-Piskunov equations, Appl. Math. Lett. 24 (8) (2011) 1428−1434.
[18] P.K. Gupta, M. Singh, Homotopy perturbation method for fractional fornberg-whitham equation, Comput. Math. Appl. 61 (2) (2011) 250−254.
[19] A.R. Seadawy, The generalized nonlinear higher order of KdV equations from the higher order nonlinear Schrodinger equation and its solutions, Optik-Int. J. Light Electron Optics. 139 (2017) 31−43.
[20] M.L. Wang, X.Z. Li, J.L. Zhang, The (G'/G)-expansion method and the traveling wave solutions to nonlinear evolution equations in mathematical physics, Phys. Lett. 372 (4) (2008) 417−423.

[21] B. Zhang, (G'/G)-expansion method for solving fractional partial differential equation in the theory of mathematical physics, Commun. Theor. Phys. 58 (2012) 623–630.
[22] B. Ayhan, A. Bekir, The (G'/G)-expansion method for the nonlinear lattice equations, Commun. Nonlinear Sci. Numer. Simul. 17 (2012) 3490–3498.
[23] M.A. Akbar, N.H.M. Ali, E.M.E. Zayed, A generalized and improved (G'/G)-expansion method for nonlinear evolution equation, Math. Probl Eng. 20 (12) (2012) 12–22.
[24] M.A. Akbar, N.H.M. Ali, E.M.E. Zayed, Abundant exact traveling wave solutions to the generalized Bretherton equation via the improved (G'/G)-expansion method, Commun. Theor. Phys. 57 (2) (2012) 173–178.
[25] S. Zhang, Q.A. Zong, D. Liu, Q. Gao, A generalized exp-function method for fractional Riccati differential equations, Commun. Fractional Calculus. 1 (1) (2010) 48–51.
[26] A. Bekir, O. Guner, A.C. Cevikel, Fractional complex transform and exp-function methods for fractional differential equations, Abstr. Appl. Anal. (2013) 426–462.
[27] M.A. Akbar, N.H.M. Ali, New solitary and periodic solutions to nonlinear evolution equation by Exp- function method, World Appl. Sci. J. 17 (12) (2012) 1603–1610.
[28] B. Lu, Backlund transformation of fractional Riccati equation and its applications to nonlinear fractional partial differential equations, Phys.Lett.A. 376 (2012) 2045–2048.
[29] S.M. Guo, L.Q. Mei, Y. Li, Y.F. Sun, The improved fractional sub-equation method and its applications to the space-time fractional differential equations in fluid mechanics, Phys. Lett. 376 (4) (2012) 407–411.
[30] S. Zhang, H.Q. Zhang, Fractional sub-equation method and its application to the nonlinear fractional PDEs, Phys. Lett. 375 (7) (2011) 1069–1073.
[31] B. Lu, The first integral method for some time fractional differential equation, J. Math. Anal. Appl. 395 (2) (2012) 684–693.
[32] A. Bekir, O. Guner, O. Unsal, The first integral method for exact solutions to nonlinear fractional differential equation, J. Comput. Nonlinear Dynam. 10 (1) (2015) 324–330.
[33] M.H. Uddin, M.A. Akbar, M.A. Khan, M.A. Haque, Close form solutions to the fractional generalized reaction Duffing model and the density dependent fractional diffusion reaction equation, Appl. Comput. Math. 6 (4) (2017) 177–184.
[34] L.X. Li, E.Q. Li, M.L. Wang, The (G'/G,$1/G$)-expansion method and its application to travelling wave solutions to the Zakharov equation, Appl. Math. B. 25 (4) (2010) 454–462.
[35] E.M.E. Zayed, M.A.M. Abdelaziz, The two variable (G'/G,$1/G$)-expansion method for solving the nonlinear KdV-mkdV equation, Math. Probl. Eng. 2012 (2012) 725061, 14 pages. https://doi.org/10.1155/2012/725061.
[36] Abdel-Salam EAB, Gumma EAE: Analytical solution of nonlinear space-time fractional differential equations using the improved fractional Riccati expansion method. Ain Shams Engr. J. doi:10.1016/j-asej.2014.10.014.
[37] M. Kaplan, A. Bekir, A. Akbulut, E. Aksoy, The modified simple equation method for nonlinear fractional differential equations, Rom. J. Phys. 60 (9) (2015) 1374–1383.
[38] R. Khalil, M. Al Horani, A. Yousef, M. Sababheh, A new definition of fractional derivative, J. Comput. Appl. Math. 264 (2014) 65–70.
[39] T.B. Benjamin, J.L. Bona, J.J. Mahony, Model equations for long waves in nonlinear dispersive systems, Phil. Trans. Roy. Soc. Lond. 272 (1972) 47–78.