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Neighborhoods and Partial Sums of a New Class of Meromorphic Multivalent Functions Defined by Fractional Calculus

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Neighborhoods and Partial Sums of a New Class of Meromorphic Multivalent Functions Defined by Fractional Calculus

Abstract

In this paper, we introduce and study a new class A^* (λ,μ,ν,η,p,ζ,τ) of meromorphic multivalent functions defined by fractional calculus operator of the punctured unit disc U^*. On this class we obtain several results like, coefficient inequality, modified Hadamard product, (N,δ)- neighborhood, partial sums, convex linear combination and integral operator.

Keywords

meromorphic multivalent function; coefficient inequality; convex linear combination; neighborhood; partial sum.

1. Introduction

Let $W^*(p)$ be the class of meromorphic functions of the form

$$f(z)=z^{-p}+\sum_{n=p}^{\infty}a_nz^n\ ,\ (n,p\in N=\{1,2,3,\dots\})\ ,$$

which are analytic and multivalent in the punctured unit disc

$U^*=\{z\in\mathbb{C}\ :0<|z|<1\}=U-\{0\}.$

We say that $f(z)$ is meromorphic multivalent starlike and meromorphic multivalent convex of order $\delta\ (0\le\delta<p,\ z\in,\ U^*f'(z)\neq0\ and\ f(z)\neq0\ )$, respectively [11] if $-Re\left\{\frac{zf'(z)}{f(z)}\right\}>\delta$ and $-\ Re\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\delta.$

Let $A^*(p)$ denotes the subclass of $W^*(p)$ containing functions of the form

$$f(z)=z^{-p}+\sum_{n=p}^{\infty}a_nz^n\ ,\ (a_n\ge0,\ n,p\in N=\{1,2,3,\dots\}) \tag{1}$$

In 2013 Atshan et al. [8] introduced the fractional differ-integral operator. They defined that operates as follows:

$$W_{0,z}^{\lambda,\mu,\nu,\eta}f(z)=\begin{cases}\dfrac{\Gamma(\mu+\nu+\eta-\lambda)\Gamma(\eta)}{\Gamma(\mu+\eta)\Gamma(\nu+\eta)}z^{-p+\eta+1}J_{0,z}^{\lambda,\mu,\nu,\eta}\left[z^{\mu+\nu}f(z)\right](0\le\lambda<1)\\ \dfrac{\Gamma(\mu+\nu+\eta-\lambda)\Gamma(\eta)}{\Gamma(\mu+\eta)\Gamma(\nu+\eta)}z^{-p+\eta+1}I_{0,z}^{\lambda,\mu,\nu,\eta}\left[z^{\mu+\nu}f(z)\right](-\infty\le\lambda<0),\end{cases} \tag{2}$$

where, $J_{0,z}^{\lambda,\mu,\nu,\eta}$ is the generalized fractional derivative operator of order λ, defined as

$$J_{0,z}^{\lambda,\mu,\nu,\eta}f(z)=\frac{1}{\Gamma(1-\lambda)}\frac{d}{dz}\left[z^{\lambda-\mu}\int_0^z t^{\eta-1}(z-t)^{\lambda}{}_2F_1\Big(\mu-\lambda,1-\nu,1-\lambda;1\ -\frac{t}{2}\Big)f(t)dt\right],$$

$$\left(0\le\lambda<1,\mu,\eta\in R,\nu\in R^+\right) \tag{3}$$

where$f$ is an analytic function on a simply-connected region of the z- plane containing the origin and multiplicity of $(z-t)^{\lambda}$ is removed by requiring $log(z-t)$ to be real when $(z-t)>0$, provided further that

$$f(z)=0(|z|^r)(z\to0),\ (r>(max\{0,\mu\}-\eta)) \tag{4}$$

and$I_{0,z}^{\lambda,\mu,\nu,\eta}$ is the generalized fractional integral operator of order $\lambda,(-\infty\le\lambda<0)$defined by

$$I_{0,z}^{\lambda,\mu,\nu,\eta}f(z)=\frac{z^{-(\lambda+\mu)}}{\Gamma(\lambda)}\left[\int_0^z t^{\eta-1}(z-t)^{\lambda-1}{}_2F_1\Big(\mu+\lambda,-\nu,\lambda;1-\frac{t}{2}\Big)f(t)\right],$$

$$\left(-\infty\le\lambda<0,\mu,\eta\in R,\nu\in R^+\right) \tag{5}$$

where$f$is constrained and the multiplicity of $(z-t)^{\lambda-1}$ is removed as above and r is given by the order estimate (4).

It follows from (3) and (5), that

$$J_{0,z}^{\lambda,\mu,\nu,1}f(z)=J_{0,z}^{\lambda,\mu,\nu}f(z)\ , \tag{6}$$

and

$$I_{0,z}^{\lambda,\mu,\nu,1}f(z)=I_{0,z}^{\lambda,\mu,\nu}f(z)\ , \tag{7}$$

where $J_{0,z}^{\lambda,\mu,\nu,1}$ and $I_{0,z}^{\lambda,\mu,\nu,1}$are the familiar Owa-Saigo-Srivastava generalized fractional derivative and integral operator respectively (see References [14–16].

Also

$$J_{0,z}^{\lambda,\lambda,\nu,1}f(z)=D_z^{\lambda}f(z)\ , \tag{8}$$

and

$$I_{0,z}^{\lambda,-\lambda,\nu,1}f(z)=D_z^{-\lambda}f(z)\ , \tag{9}$$

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where D_z^{λ} and $D_z^{-\lambda}$ are the familiar Owa-Srivastava fractional derivative and integral of order λ, respectively (see, Owa [13], Srivastava and Owa [14]).

Furthermore, in terms of Gamma function, we have

$$J_{0,z}^{\lambda,\mu,\nu,\eta} z^k = \frac{\Gamma(k+\eta)\Gamma(k+\eta-\mu+\nu)}{\Gamma(k+\eta-\mu)\Gamma(k+\eta+\lambda+\nu)} z^{k+\eta-\mu-1}, \tag{10}$$

$\left(0 \le \lambda < 1, \mu, \eta \in R, \nu \in R^+\right)$ *and* $(k > (max\{0,\mu\} - \eta))$,

and

$$I_{0,z}^{\lambda,\mu,\nu,\eta} z^k = \frac{\Gamma(k+\eta)\Gamma(k+\eta-\mu+\nu)}{\Gamma(k+\eta-\mu)\Gamma(k+\eta-\lambda+\nu)} z^{k+\eta-\mu-1}, \tag{11}$$

$\left(-\infty \le \lambda < 0, \mu, \eta \in R, \nu \in R^+\right)$ *and* $(k > (max\{0,\mu\} - \eta))$.

Now by using (1), (10) and (11) in (2), we find that

$$W_{0,z}^{\lambda,\mu,\nu,\eta} f(z) = z^{-p} + \sum_{n=p}^{\infty} \Gamma_n^{\lambda,\mu,\nu,\eta} a_n z^n, \tag{12}$$

Provided that $-\infty < \lambda < 1,\ \mu+\nu+\eta > \lambda,\ \mu > -\eta,\ \eta > 0,\ p \in N,\ f \in W^*(p)$ and

$$\Gamma_n^{\lambda,\mu,\nu,\eta} = \frac{(\mu+\eta)_{n+p}(\nu+\eta)_{n+p}}{(\mu+\nu+\eta-\lambda)_{n+p}(\eta)_{n+p}}.$$

It may be worth noting that, by choosing $\mu = \lambda,\ \eta = 1$ and $p = 1$, the operator $W_{0,z}^{\lambda,\mu,\nu,\eta} f(z)$ reduces to the Ruscheweyh derivative $D^{\lambda}F(Z)$ for the meromorphic univalent function [18].

Next, we define the new class of meromorphic multivalent functions by giving the condition for the function f which defined in (1) belongs to the class $A^*(a,c,k,\beta,\alpha,\gamma,\mu)$, by using (11).

Definition 1. A function $f \in A^*(p)$ of the form (1) is said to be in the class $A^*(\lambda,\mu,\nu,\eta,\ p,\zeta,\tau)$, if satisfies the following condition:

$$\left| \frac{\dfrac{z^2\left(W_{0,z}^{\lambda,\mu,\nu,\eta} f(z)\right)'' + z(p+1)\left(W_{0,z}^{\lambda,\mu,\nu,\eta} f(z)\right)'}{W_{0,z}^{\lambda,\mu,\nu,\eta} f(z)} + 1}{\dfrac{z^2\left(W_{0,z}^{\lambda,\mu,\nu,\eta} f(z)\right)'' + z\left(W_{0,z}^{\lambda,\mu,\nu,\eta} f(z)\right)'}{W_{0,z}^{\lambda,\mu,\nu,\eta} f(z)} + \zeta} \right| \le \tau, \tag{13}$$

where $\left(\zeta \ge 1,\ \frac{1}{2} < \tau \le 1,\ \ p = 1,2,3,\ \ldots\right)$ *and* $(-\infty < \lambda < 1,\ \mu+\nu+\eta > \lambda,\ \mu > -\eta,\ \eta > 0)$.

Some other authors were studies different classes of meromorphic multivalent functions with other operators like Aouf [2], Aouf and Shammaky [7], Raina and Srivastava [17] and Atshan and Kulkarni [10], see also References [3–6].

In this paper, we study and discuss the new class $A^*(\lambda,\mu,\nu,\eta,\ p,\zeta,\tau)$ of meromorphic multivalent functions defined by fractional calculus operator in the punctured unit disc U*. We get several properties like coefficient inequality, neighborhood system, partial sum, extreme point and obtain some interesting results.

2. Coefficient estimates

In our first theorem, we give necessary and sufficient condition for a function f to be in the class $A^*(\lambda,\mu,\nu,\eta,\ p,\zeta,\tau)$.

Theorem 1. *A function $f(z)$ defined by (1) is in the class $A^*(\lambda,\mu,\nu,\eta,\ p,\zeta,\tau)$. if and only if*

$$\sum_{n=p}^{\infty} n[n+p-\tau(n+\zeta)-1]\Gamma_n^{\lambda,\mu,\nu,\eta} a_n \le \tau\left(p^2+\zeta\right) - 1, \tag{14}$$

where $\left(\zeta \ge 1,\ \frac{1}{2} < \tau \le 1,\ \ p,n = 1,2,3,\ldots\right)$ *and* $(-\infty < \lambda < 1,\ \mu+\nu+\eta > \lambda,\ \mu > -\eta,\ \eta > 0)$.

Proof. *From (13) we have*

$$\left| \frac{z^2\left(W_{0,z}^{\lambda,\mu,\nu,\eta} f(z)\right)'' + z(p+1)\left(W_{0,z}^{\lambda,\mu,\nu,\eta} f(z)\right)' + W_{0,z}^{\lambda,\mu,\nu,\eta} f(z)}{z^2\left(W_{0,z}^{\lambda,\mu,\nu,\eta} f(z)\right)'' + z\left(W_{0,z}^{\lambda,\mu,\nu,\eta} f(z)\right)' + \zeta W_{0,z}^{\lambda,\mu,\nu,\eta} f(z)} \right| \le \tau.$$

Assume that (14) holds. It is sufficient to show that

$$M = \left| z^2\left(W_{0,z}^{\lambda,\mu,\nu,\eta} f(z)\right)'' + z(p+1)\left(W_{0,z}^{\lambda,\mu,\nu,\eta} f(z)\right)' + W_{0,z}^{\lambda,\mu,\nu,\eta} f(z) \right| - \tau \left| z^2\left(W_{0,z}^{\lambda,\mu,\nu,\eta} f(z)\right)'' + z\left(W_{0,z}^{\lambda,\mu,\nu,\eta} f(z)\right)' + \zeta W_{0,z}^{\lambda,\mu,\nu,\eta} f(z) \right| < 0$$

$$= \Bigg| z^2 \Bigg[p(p+1)z^{-p-2} + n(n-1)\sum_{n=p}^{\infty} \Gamma_n^{\lambda,\mu,\nu,\eta} a_n z^{n-2} \Bigg] + z(p+1)\Bigg[-pz^{-p-1} + n\sum_{n=p}^{\infty} \Gamma_n^{\lambda,\mu,\nu,\eta} a_n z^{n-1} \Bigg] + \Bigg[z^{-p} + n\sum_{n=p}^{\infty} \Gamma_n^{\lambda,\mu,\nu,\eta} a_n z^{n-1} \Bigg] \Bigg|$$

$$- \tau \Bigg| z^2 \Bigg[p(p+1)z^{-p-2} + n(n-1)\sum_{n=p}^{\infty} \Gamma_n^{\lambda,\mu,\nu,\eta} a_n z^{n-2} \Bigg] + z\Bigg[-pz^{-p-1} + n\sum_{n=p}^{\infty} \Gamma_n^{\lambda,\mu,\nu,\eta} a_n z^{n-1} \Bigg] + \zeta\Bigg[z^{-p} + n\sum_{n=p}^{\infty} \Gamma_n^{\lambda,\mu,\nu,\eta} a_n z^{n-1} \Bigg] \Bigg|$$

for $|z| = r < 1$, from (13), *we get*

$$M = \Bigg| \Bigg[z^{-p} + \sum_{n=p}^{\infty} n[n+p-1]\Gamma_n^{\lambda,\mu,\nu,\eta} a_n z^n \Bigg] \Bigg| - \tau \Bigg| \Bigg[(p^2+\zeta) z^{-p} + \sum_{n=p}^{\infty} n[n+\zeta]\Gamma_n^{\lambda,\mu,\nu,\eta} a_n z^n \Bigg] \Bigg|$$

$$\le r^{-p} + \sum_{n=p}^{\infty} n\left[n+p-1\Gamma_n^{\lambda,\mu,\nu,\eta}\right] a_n r^n - \tau\left(p^2+\zeta\right) z^{-p} - \tau \sum_{n=p}^{\infty} n[n+\zeta]\Gamma_n^{\lambda,\mu,\nu,\eta} a_n z^n$$

$$< \sum_{n=p}^{\infty} n[n+p-\tau(n+\zeta)-1]\Gamma_n^{\lambda,\mu,\nu,\eta} a_n - \tau\left(p^2+\zeta\right) + 1 \le 0,$$

By hypothesis

$$\sum_{n=p}^{\infty} n[n+p-\tau(n+\zeta)-1]\Gamma_n^{\lambda,\mu,\nu,\eta} a_n \le \tau\left(p^2+\zeta\right) - 1 .$$

Hence $f \in A^*(\lambda,\mu,\nu,\eta,\ p,\zeta,\tau)$.

Conversely, let $f(z) \in A^*(\lambda,\mu,\nu,\eta,\ p,\zeta,\tau)$. Then (13) holds so we have

$$\left| \frac{z^2\left(W_{0,z}^{\lambda,\mu,\nu,\eta} f(z)\right)'' + z(p+1)\left(W_{0,z}^{\lambda,\mu,\nu,\eta} f(z)\right)' - W_{0,z}^{\lambda,\mu,\nu,\eta} f(z)}{z^2\left(W_{0,z}^{\lambda,\mu,\nu,\eta} f(z)\right)'' + z\left(W_{0,z}^{\lambda,\mu,\nu,\eta} f(z)\right)' + \zeta W_{0,z}^{\lambda,\mu,\nu,\eta} f(z)} \right|$$

$$= \left| \frac{z^{-p} + \sum_{n=p}^{\infty} n[n+p-1]\Gamma_n^{\lambda,\mu,\nu,\eta} a_n z^n}{(p^2+\zeta) z^{-p} + \sum_{n=p}^{\infty} n[n+\zeta]\Gamma_n^{\lambda,\mu,\nu,\eta} a_n z^n} \right| \le \tau.$$

.

Since $Re(z) \le |z|$ for all z, it follows that

$$Re\left\{ \frac{z^{-p} + \sum_{n=p}^{\infty} n[n+p-1]\Gamma_n^{\lambda,\mu,\nu,\eta} a_n z^n}{(p^2+\zeta) z^{-p} + \sum_{n=p}^{\infty} n[n+\zeta]\Gamma_n^{\lambda,\mu,\nu,\eta} a_n z^n} \right\} \le \tau .$$

Now, if z is real we have $W_{0,z}^{\lambda,\mu,\nu,\eta} f(z)$ is real.

Letting $z \to 1^-$ through real values, we obtain

$$z^{-p} + \sum_{n=p}^{\infty} n[n+p-1]\Gamma_n^{\lambda,\mu,\nu,\eta} a_n z^n \le \tau\left[(p^2+\zeta) z^{-p} + \sum_{n=p}^{\infty} n[n+\zeta]\Gamma_n^{\lambda,\mu,\nu,\eta} a_n z^n \right].$$

From the last inequality, we have

$$\sum_{n=p}^{\infty} n[n+p-\tau(n+\zeta)-1]\Gamma_n^{\lambda,\mu,\nu,\eta} a_n \le \tau\left(p^2+\zeta\right) - 1 .$$

This completes the Proof

Finally, sharpness follows if we take

$$f(z) = \frac{1}{z^p} + \frac{\tau\left(p^2+\zeta\right) - 1}{n[n+p-\tau(n+\zeta)-1]\Gamma_n^{\lambda,\mu,\nu,\eta}} z^n, \quad (n,p = 1,2,\ldots) \tag{15}$$

Corollary 1. *If* $f(z)$ *defined by* (1) *is in the class* $A^*(\lambda,\mu,\nu,\eta,\ p,\zeta,\tau)$, *then*

$$a_n \le \frac{\tau\left(p^2+\zeta\right) - 1}{n[n+p-\tau(n+\zeta)-1]\Gamma_n^{\lambda,\mu,\nu,\eta}}, \tag{16}$$

where $\left(\zeta \ge 1,\ \frac{1}{2} < \tau \le 1,\ p,n = 1,2,3,\ldots\right)$ *and* $(-\infty < \lambda < 1,\ \mu+\nu+\eta > \lambda,\ \mu > -\eta,\ \eta > 0)$.

3. Neighborhood properties

The earlier works on neighborhoods of analytic functions by Goodman [11]and Ruscheweyh [19] for the elements of several famous subclasses of analytic functions. Altintas and Owa [1] considered for a certain family of analytic functions with a negative coefficient. Also, Liu and Srivastava [12] and Atshan [9] extend for a certain subclass of meromorphically univalent and multivalent functions. In this paper we define the (n,ϵ)- neighborhood of a function $f \in w^*(z)$ as follows:

$$N_{n,\epsilon}(f) = \left\{ q \in A^*(p) : q(z) = z^{-p} + \sum_{n=p}^{\infty} a_n z^n \text{ and } \sum_{n=p}^{\infty} n|a_n - b_n| \leq \epsilon \,; 0 \leq \epsilon < 1 \right\} \tag{17}$$

For the identity function $e(z) = z$, we have

$$N_{n,\epsilon}(e) = \left\{ q \in A^*(p) : q(z) = z^{-p} + \sum_{n=p}^{\infty} a_n z^n \text{ and } \sum_{n=p}^{\infty} n|b_n| \leq \epsilon \quad ; 0 \leq \epsilon < 1 \right\} \tag{18}$$

Definition 2. *A function $f \in A^*(p)$ is said to be in the class $A^*(\lambda,\mu,\nu,\eta,\ p,\zeta,\tau)$, if there exists a function $q \in A^*(\lambda,\mu,\nu,\eta,\ p,\zeta,\tau)$ such that*

$$\left|\frac{f(z)}{q(z)} - 1\right| < 1 - \delta, \quad \left(z \in U^*,\ 0 \leq \delta < 1\right).$$

Theorem 2. *If $q \in A^*(\lambda,\mu,\nu,\eta,\ p,\zeta,\tau)$, and*

$$\delta = 1 - \frac{\epsilon p[2p - \tau(n+\zeta) - 1]\Gamma_p^{\lambda,\mu,\nu,\eta}}{p^2[2p - \tau(n+\zeta) - 1]\Gamma_p^{\lambda,\mu,\nu,\eta} - \tau p\,(p^2+\zeta) + 1}, \tag{19}$$

Then $N_{n,\epsilon}(q) \subset A_\delta^*(\lambda,\mu,\nu,\eta,\ p,\zeta,\tau)$.

Proof. Let $f \in N_{n,\epsilon}(q)$. Then by (16) we get

$$\sum_{n=p}^{\infty} n|a_n - b_n| \leq \epsilon\,,$$

Which implies the coefficient inequality

$$\sum_{n=p}^{\infty} |a_n - b_n| \leq \frac{\epsilon}{p}(n \in N).$$

Since $q \in A^*(\lambda,\mu,\nu,\eta,\ p,\zeta,\tau)$, we have from Theorem (1), that,

$$\sum_{n=p}^{\infty} b_n \leq \frac{\tau\,(p^2+\zeta) - 1}{p[2p - \tau(n+\zeta) - 1]\Gamma_p^{\lambda,\mu,\nu,\eta}}.$$

So that

$$\left|\frac{f(z)}{q(z)} - 1\right| \leq \frac{\sum_{n=p}^{\infty}|a_n - b_n|}{1 - \sum_{n=p}^{\infty} b_n} \leq \frac{\epsilon p[2p - \tau(n+\zeta) - 1]\Gamma_p^{\lambda,\mu,\nu,\eta}}{p^2[2p - \tau(n+\zeta) - 1]\Gamma_p^{\lambda,\mu,\nu,\eta} - \tau p\,(p^2+\zeta) + 1}$$

$$= 1 - \delta\,.$$

Therefore, by Definition (2),$f \in A_\delta^*(\lambda,\mu,\nu,\eta,\ p,\zeta,\tau)$, for δ given by (19)

Theorem 3. If $q \in A^*(\lambda,\mu,\nu,\eta,\ p,\zeta,\tau)$, and

$$\epsilon = \frac{\tau\,(p^2+\zeta) - 1}{[2p - \tau(p+\zeta) - 1]},$$

Then $A^*(\lambda,\mu,\nu,\eta,\ p,\zeta,\tau) \subset N_{n,\epsilon}(e)$.

Proof. Since $q \in A^*(\lambda,\mu,\nu,\eta,\ p,\zeta,\tau)$, we have from Theorem (1), that,

$$p[2p - \tau(p+\zeta) - 1]\Gamma_p^{\lambda,\mu,\nu,\eta} \sum_{n=p}^{\infty} a_n \leq \tau\,(p^2+\zeta) - 1\,. \tag{20}$$

On the other hand from (13) and (19), we have

$$p(2p-1)\sum_{n=p}^{\infty}\Gamma_n^{\lambda,\mu,\nu,\eta}a_n \le \left[\tau\left(p^2+\zeta\right)-1\right] + \tau p(p+\zeta)\Gamma_p^{\lambda,\mu,\nu,\eta}\sum_{n=p}^{\infty}a_n$$

$$\le \left[\tau\left(p^2+\zeta\right)-1\right] + \tau p(p+\zeta)\Gamma_p^{\lambda,\mu,\nu,\eta}\frac{\tau\left(p^2+\zeta\right)-1}{p[2p-\tau(p+\zeta)-1]\Gamma_p^{\lambda,\mu,\nu,\eta}}\,.$$

Or equivalent to

$$\sum_{n=p}^{\infty}\Gamma_n^{\lambda,\mu,\nu,\eta}na_n = \frac{\tau\left(p^2+\zeta\right)-1}{[2p-\tau(p+\zeta)-1]} = \epsilon\,.$$

Hence, by (17) we get $q\in N_{n,\epsilon}(e)$. That is $A^*(\lambda,\mu,\nu,\eta,p,\zeta,\tau)\subset N_{n,\epsilon}(e)$

4. Partial sum

Theorem 4. *Let $f\in A^*(\lambda,\mu,\nu,\eta,p,\zeta,\tau)$ be given in (1). Define the partial sums $S_1(z)$ and $S_m(z)$ as*

$$S_1(z)=z^{-p} \text{ and } S_m(z)=z^{-p}+\sum_{n=p}^{m-1}a_nz^n\,,\ \left(m\in\mathbb{N}\setminus\{1\},\ z\in U^*\right).$$

Suppose that

$$\sum_{n=p}^{\infty}d_na_n\le 1\,,\ \left(d_n=\frac{n[n+p-\tau(n+\zeta)-1]\Gamma_n^{\lambda,\mu,\nu,\eta}}{\tau\left(p^2+\zeta\right)-1}\right). \tag{21}$$

Then,

$$Re\left\{\frac{f(z)}{s_m(z)}\right\}>1-\frac{1}{d_m}\,,\quad \left(z\in U^*\,,\ m\in\mathbb{N}\right), \tag{22}$$

and

$$Re\left\{\frac{s_m(z)}{f(z)}\right\}>\frac{d_m}{1+d_m}\,.\qquad \left(z\in U^*\,,\ m\in\mathbb{N}\right) \tag{23}$$

Proof. We note that from (21) $d_{n+1}>d_n>1$ $(n\in\mathbb{N})$, so$\{d_n\}(n\in\mathbb{N})$is an increasing sequence, therefore

$$\sum_{n=p}^{m-1}a_n+d_m\sum_{n=m}^{\infty}a_n\le\sum_{n=p}^{\infty}d_na_n\le 1\,. \tag{24}$$

By setting

$$q_1(z)=d_m\left[\frac{f(z)}{s_m(z)}-\left(1-\frac{1}{d_m}\right)\right]=1+\frac{d_m\sum_{n=m}^{\infty}a_nz^n}{1+\sum_{n=p}^{m-1}a_nz^n}\,,$$

from (24), we find that

$$\left|\frac{q_1(z)-1}{q_1(z)+1}\right|\le\frac{d_m\sum_{n=m}^{\infty}a_n}{2-2\sum_{n=p}^{m-1}a_n-d_m\sum_{n=m}^{\infty}a_n}\le 1\,.\quad \left(z\in U^*\right)$$

Therefore, $Re\{q_1(z)\}>0$, and (22) satisfied.

Similarly, if we let

$$q_2(z)=(1+d_m)\left[\frac{s_m(z)}{f(z)}-\frac{d_m}{1+d_m}\right] = 1-\frac{(1+d_m)\sum_{n=m}^{\infty}a_nz^n}{1+\sum_{n=p}^{m-1}a_nz^n}\,,$$

By using (24), *we have*

$$\left|\frac{q_2(z)-1}{q_2(z)+1}\right|\le\frac{(1+d_m)\sum_{n=m}^{\infty}a_n}{2+(1-d_m)\sum_{n=m}^{\infty}a_n-2\sum_{n=1}^{m-1}a_n}\le 1\,,\ \left(z\in U^*\right)$$

and this show that $Re\{q_2(z)\}>0$, *so* (23) is satisfied

Now , we define the function$f_i(z)(i=1,2,)$, as follows

$$f_i(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_{n,i}z^n\ (a_{n,i}\ge 0\,,\ n\in\ \mathbb{N}), \tag{25}$$

and use it in the next three theorems.

5. Modified Hadamard product

Theorem 5. *Let the function $f_i(z)(i=1,2,)$, defined in (23) be in the class $A^*(\lambda,\mu,\nu,\eta,p,\zeta,\tau)$, then $f_1(z)*f_2(z)\in A^*(\lambda,\mu,\nu,\eta,p,\zeta,\delta)$, where*

$$\delta \geq \frac{(n+p-1)[\tau(p^2+\zeta)-1]^2 + n[n+p-\tau(n+\zeta)-1]^2 \Gamma_n^{\lambda,\mu,\nu,\eta}}{n(p^2+\zeta)[n+p-\tau(n+\zeta)-1]^2 \Gamma_n^{\lambda,\mu,\nu,\eta} + (n+\zeta)[\tau(p^2+\zeta)-1]^2} .$$

Proof. It is sufficient to find the smallestδ, such that

$$\sum_{n=p}^{\infty} \frac{n[n+p-\delta(n+\zeta)-1]\Gamma_n^{\lambda,\mu,\nu,\eta}}{\delta(p^2+\zeta)-1} a_{n,1}a_{n,2} \leq 1 ,$$

Since $f_i \in A^*(\lambda,\mu,\nu,\eta,\ p,\zeta,\tau)$, $(i=1,2,)$, then

$$\sum_{n=p}^{\infty} \frac{n[n+p-\tau(n+\zeta)-1]\Gamma_n^{\lambda,\mu,\nu,\eta}}{\tau(p^2+\zeta)-1} a_{n,1}a_{n,2} \leq 1 , \tag{26}$$

By Cauchy-Schwarz inequality, we get

$$\sum_{n=p}^{\infty} \frac{n[n+p-\tau(n+\zeta)-1]\Gamma_n^{\lambda,\mu,\nu,\eta}}{\tau(p^2+\zeta)-1} \sqrt{a_{n,1}a_{n,2}} \leq 1. \tag{27}$$

We want only to show that

$$\sum_{n=p}^{\infty} \frac{n[n+p-\delta(n+\zeta)-1]\Gamma_n^{\lambda,\mu,\nu,\eta}}{\delta(p^2+\zeta)-1} a_{n,1}a_{n,2} \leq \sum_{n=p}^{\infty} \frac{n[n+p-\tau(n+\zeta)-1]\Gamma_n^{\lambda,\mu,\nu,\eta}}{\tau(p^2+\zeta)-1} \sqrt{a_{n,1}a_{n,2}} .$$

This equivalent to

$$\frac{n[n+p-\delta(n+\zeta)-1]}{\delta(p^2+\zeta)-1} \sqrt{a_{n,1}a_{n,2}} \leq \frac{n[n+p-\tau(n+\zeta)-1]}{\tau(p^2+\zeta)-1} .$$

Or

$$\sqrt{a_{n,1}a_{n,2}} \leq \frac{n[n+p-\tau(n+\zeta)-1][\delta(p^2+\zeta)-1]}{n[\tau(p^2+\zeta)-1][n+p-\delta(n+\zeta)-1]} .$$

From (27), we have

$$\sqrt{a_{n,1}a_{n,2}} \leq \frac{\tau(p^2+\zeta)-1}{n[n+p-\tau(n+\zeta)-1]\Gamma_n^{\lambda,\mu,\nu,\eta}} .$$

Therefore, it is sufficient to show that

$$\frac{\tau(p^2+\zeta)-1}{n[n+p-\tau(n+\zeta)-1]\Gamma_n^{\lambda,\mu,\nu,\eta}} \leq \frac{n[n+p-\tau(n+\zeta)-1][\delta(p^2+\zeta)-1]}{n[\tau(p^2+\zeta)-1][n+p-\delta(n+\zeta)-1]} .$$

Which implies

$$\delta \geq \frac{(n+p-1)[\tau(p^2+\zeta)-1]^2 + n[n+p-\tau(n+\zeta)-1]^2 \Gamma_n^{\lambda,\mu,\nu,\eta}}{n(p^2+\zeta)[n+p-\tau(n+\zeta)-1]^2 \Gamma_n^{\lambda,\mu,\nu,\eta} + (n+\zeta)[\tau(p^2+\zeta)-1]^2}$$

Theorem 6. *Let the function $f_i(z)(i=1,2,)$, defined in (25) be in the class $A^*(\lambda,\mu,\nu,\eta,\ p,\zeta,\tau)$, then $f_1(z) * f_2(z) \in A^*(\lambda,\mu,\nu,\eta,\ p,\zeta,\tau)$.*

Proof. Because $f_1(z)$is a member in the class $A^*(\lambda,\mu,\nu,\eta,\ p,\zeta,\tau)$, then by Theorem(1), we have

$$\sum_{n=p}^{\infty} n[n+p-\tau(n+\zeta)-1]\Gamma_n^{\lambda,\mu,\nu,\eta} a_{n,1} \leq \tau\left(p^2+\zeta\right) - 1 .$$

Since

$$\sum_{n=p}^{\infty} \frac{n[n+p-\tau(n+\zeta)-1]\Gamma_n^{\lambda,\mu,\nu,\eta}}{\tau\,(p^2+\zeta)-1} |a_{n,1}a_{n,2}| = \sum_{n=p}^{\infty} \frac{n[n+p-\tau(n+\zeta)-1]\Gamma_n^{\lambda,\mu,\nu,\eta}}{\tau\,(p^2+\zeta)-1} a_{n,1}|a_{n,2}| \leq \sum_{n=p}^{\infty} \frac{n[n+p-\tau(n+\zeta)-1]\Gamma_n^{\lambda,\mu,\nu,\eta}}{\tau\,(p^2+\zeta)-1} a_{n,1} \leq 1 .$$

Therefore $f_1(z) * f_2(z) \in A^*(\lambda,\mu,\nu,\eta,\ p,\zeta,\tau)$

6. Convex linear combination

Theorem 7. *The class $A^*(\lambda,\mu,\nu,\eta,\ p,\zeta,\tau)$, is closed under convex linear combination.*

Proof. Let$f_i(z)(i = 1,2,\ )$, be a function defined in (25) be in the class $A^*(\lambda,\mu,\nu,\eta,\ p,\zeta,\tau)$, we want to show the function $h(z) = \beta f_1(z) + (1-\beta) f_2(z)$, $(0 \le \beta \le 1)$ is also in the class $A^*(\lambda,\mu,\nu,\eta,\ p,\zeta,\tau)$. Since for $0 \le \beta \le 1$.

$$h(z) = z^{-p} + \sum_{n=p}^{\infty} [\beta a_{n,1} + (1-\beta) a_{n,2}] z^n,$$

Then by Theorem (1), we have

$$\sum_{n=p}^{\infty} n[n+p-\tau(n+\zeta)-1]\Gamma_n^{\lambda,\mu,\nu,\eta}[\beta a_{n,1} + (1-\beta)a_{n,2}]$$

$$= \beta \sum_{n=p}^{\infty} n[n+p-\tau(n+\zeta)-1]\Gamma_n^{\lambda,\mu,\nu,\eta} a_{n,1}$$

$$+ (1-\beta)\sum_{n=p}^{\infty} n[n+p-\tau(n+\zeta)-1]\Gamma_n^{\lambda,\mu,\nu,\eta} a_{n,2}$$

$$\le \beta\left[\tau\left(p^2+\zeta\right)-1\right] + (1-\beta)\left[\tau\left(p^2+\zeta\right)-1\right]$$

$$= \tau\left(p^2+\zeta\right) - 1.$$

Therefore by Theorem (1) $h(z) \in A^*(\lambda,\mu,\nu,\eta,\ p,\zeta,\tau)$

7. Integral operator

Theorem 8. *Let the function f given by (1) be in the class $A^*(\lambda,\mu,\nu,\eta,\ p,\zeta,\tau)$. Then the integral operator*

$$F_\delta(Z) = (1+\delta)z^{-p} + \delta p \int_0^z \frac{f(s)}{s}\, ds,\ \left(\delta \ge 0,\ z \in U^*\right),$$

in the class $A^*(a,c,k,\beta,\alpha,\gamma,\delta)$, if $0 \le \delta \le \frac{1}{p}$.

Proof. According to (1)

$$f(z) = z^{-p} + \sum_{n=p}^{\infty} a_n z^n,$$

Then

$$F_\delta(Z) = (1+\delta)z^{-p} + \delta p \int_0^z \left(\frac{s^{-p} + \sum_{n=p}^{\infty} a_n s^n}{s}\right) ds.$$

$$= (1+\delta)z^{-p} + \delta p \left(\frac{-1}{p} z^{-p} + \sum_{n=p}^{\infty} \frac{a_n}{n} z^n\right)$$

$$\le z^{-p} + \sum_{n=p}^{\infty} \frac{\delta p a_n}{n} z^n$$

$$= z^{-p} + \sum_{n=p}^{\infty} g_n z^n, \text{ where} g_n = \frac{\delta p a_n}{n}$$

Now, if we consider that

$$\sum_{n=p}^{\infty} n[n+p-\tau(n+\zeta)-1]\Gamma_n^{\lambda,\mu,\nu,\eta} g_n$$

$$= \sum_{n=p}^{\infty} n[n+p-\tau(n+\zeta)-1]\Gamma_n^{\lambda,\mu,\nu,\eta} \frac{\delta p a_n}{n}$$

$$\le \sum_{n=p}^{\infty} n[n+p-\tau(n+\zeta)-1]\Gamma_n^{\lambda,\mu,\nu,\eta} \delta p a_n$$

$$\le \sum_{n=p}^{\infty} n[n+p-\tau(n+\zeta)$$
$$-1]\Gamma_n^{\lambda,\mu,\nu,\eta} a_n, \quad \textit{since}\ (\delta p \le 1)$$

$$\le \tau\left(p^2+\zeta\right) - 1.$$

Therefore by Theorem (1), we have $F_\delta(Z)$belongs to $A^*(a,c,k,\beta,\alpha,\gamma,\delta)$

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References

[1] O. Altintas, S. Owa, Neighborhood of certain analytic functions with negative coefficient, Int. J. Math. Sci. 19 (1996) 797–800.
[2] M.K. Aouf, A family of meromorphically P-valent functions with positive coefficient, Gen. Math. 20 (1) (2012) 25–37.
[3] A.K. AL-khafaji, W.G. Atshan, S.S. Abed, On the generalization of a class of harmonic univalent functions defined by differential operator, Mathematics 6 (12) (2018) 312.
[4] A.K. AL-khafaji, W.G. Atshan, S.S. Abed, On a differential subordination of a certain subclass of univalent functions, J. Kufa. Math. Comp. 5 (3) (2018) 11–16.
[5] AL-khafaji, A. K.; Atshan, W. G.; Abed, S. S. Some properties of a class of harmonic multivalent functions.Accepted in J. Univ. Babylon, Pure Appl. Sci.
[6] AL-khafaji, A. K.; Atshan, W. G.; Abed, S. S. Some interesting properties of a novel subclass of multivalent functions. Accepted in J. Kufa. Math. Comp..
[7] M.K. Aouf, A.E. Shamaky, A certain subclass of meromorphically convex functions with negative coefficient, J. Approx. Theory Appl. 12 (2005) 123–143.

[8] W.,G. Atshan, L.A. Alzopee, M. Mostafa, On fractional calculus operators of a class of meromorphic multivalent functions, Gen. Math. Notes 18 (2) (2013) 92−103.
[9] W.G. Atshan, Subclass of meromorphic functions with positive coefficients defined by Ruscheweyh derivative II, J. Surv. Math. Its Appl. 3 (2008) 67−77.
[10] W.G. Atshan, S.R. Kulkarni, Subclass of meromorphic functions with positive coefficient defined by Ruscheweyh derivative-I, J. Rajasthan Acad. Phys. Sci. 6 (2) (2007) 129−140.
[11] A.W. Goodman, Univalent function and non-analytic curves, Proc. Am. Math. Soc. 8 (1975) 598−601.
[12] J.L. Liu, H.M. Srivastava, A linear operator and associated families of meromorficlly multivalent functions, J. Math. Anal. Appl. 259 (2001) 566−581.
[13] S. Owa, On the distortion the theorems-I, Kyungpook Math. J. 18 (1978) 53−59.
[14] S. Owa, H.M. Srivastava, Current to Piscine Analytic Functions Theory, World Scientific Publishing Company, Singapore, New Jersey. London And Hong Kong, 1992.
[15] S. Owa, M. Saigo, H.M. Srivastava, A class of distortion theorems involving certain operator of fractional calculus, J. Math. Anal. Appl. 31 (1988) 412−420.
[16] S. Owa, M. Saigo, H.M. Srivastava, Some characterization theorems for starlike and convex functions involving a certain fractional integral operator, J. Math. Anal. Appl. 140 (1989) 419−426.
[17] R.K. Raina, H.M. Srivastava, A new class of merophically multivalent functions with applications general hypergeomatric functions, Math. Comput. Model. 213 (2006) 350−356.
[18] S. Ruscheweyh, New certain for univalent functions, Proc. Am. Math. Soc. 49 (1975) 109−115.
[19] S. Ruscheweyh, Neighborhood of univalent functions, Proc. Am. Math. Soc. 81 (1981) 521−527.