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Derivation of Direct Explicit Integrators of RK Type for Solving Class of Seventh-Order Ordinary Differential Equations

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Abstract

The main contribution of this work is the development of direct explicit methods of Runge-Kutta (RK) type for solving class of seventh-order ordinary differential equations (ODEs) to improve computational efficiency. For this purpose, we have generalized RK, RKN, RKD, RKT, RKFD and RKM methods for solving class of first-, second-, third-, fourth-, and fifth-order ODEs. Using Taylor expansion approach, we have derived the algebraic equations of the order conditions for the proposed RKM integrators up to the tenth-order. Based on these order conditions, two RKM methods of fifth- and sixth-order with four- and five-stage are derived. The zero stability of the methods is proven. Stability polynomial of the methods for linear special seventh-order ODE is given. Numerical results have clearly shown the advantage and the efficiency of the new methods and agree well with analytical solutions due to the fact the proposed integrators are zero stable, more efficient and accurate integrators.

Keywords

Runge-Kutta method (RK); Runge-Kutta Nystrom method (RKN); Direct method (RKD); RKM; RKT; RKFD; Integrators; Class of seventh-order; Stage; Ordinary differential equations; Order conditions; Taylor expansion.

Cover Page Footnote

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1. Introduction

The mathematical modeling of many real-life problems in physics, engineering and economics can be written as higher-order differential equations (DEs) ; ordinary or partial, DEs are important tools for mathematical models in quantum mechanics and nonlinear optics. Typical examples can be found in different fields such as quantum physics, solid state physics and fluid physics plasma physics [1]. For example, the seventh-order Korteweg-de Vrie (KdV) is a nonlinear PDE, another application of seventh-order DE is nonlinear dispersive equations, which include several models arising in the study of different physical phenomena. For the review of the researches in seventh-order DEs [2], has studied an efficient numerical solution for seventh-order DEs by using septic B-spline collocation method while [3] has proposed a new method based on the Lengendre wavelets expansion together with operational matrices of fractional integration and derivative to solve time-fractional seventh-order KdV equation (sKdV) [4] has used differential transformation method for solving seventh-order BVPs [5] has solved the seventh -order ODEs by Haar wavelet approach [6] is performed Lie symmetry analysis of the seventh-order time fractional SawadaKoteralto (FSKI) equation with Remain-Liouville derivative [7], is employed the variational iteration method using Hes polynomials to solve the seventh-order boundary value problems (BVPs) and [8] is solved the Laxs seventh-order Korteweg-de Vires (KdV) equation by pseudospectral method. Moreover [17], is presented the numerical computations for water-based nanofluids with A12O3 and Cu nanopar-ticles and [9] has introduced the numerical solution of a thermal instability problem in a rotating nanofluid layer [10,11] have derived direct explicit integrators for solving third-order ODEs. Furthermore [12,13], have introduced direct explicit integrators for solving fourth- and fifth- order ODEs respectively. However, to improve the computational efficiency of the numerical methods for solving class of seventh-order ODEs, the indirect numerical methods improved to be direct methods. Using Taylor expansion approach, we have derived the algebraic equations of the order conditions for the proposed RKM integrators up to the tenth-order. Based on these order conditions, two RKM methods of fifth- and sixth-order with four-and five-stage are derived. The novelty of this work is the generalization RK, RKN, RKD, RKT, RKFD and RKM methods for solving class of first-, second-, third-, fourth-, and fifth-order ODEs. Numerical implementations using Maple and MATLAB show that the numerical solutions of two proposed integrators agree well with analytical solutions due to the fact the proposed integrators are efficient and accurate and have less computational time comparing with indirect methods.

2. Preliminary

Here, we give the definition for a class of general quasi linear seventh-order ODE as the follows:

2.1. Class of seventh-order ordinary differential equations

In this study, we concerned with class of seventh-order ODEs with no appearance for the derivatives up to sixth order. It can be written in the following form:

$$\mathscr{Y}^{(7)}(t) = f(t, \mathscr{Y}(t)); t \geq t_0 \tag{1}$$

Subject to initial condition: $\mathscr{Y}(t_0) = \overline{a_i}; i = 0, 1, \ldots, 6$.

where

$$f : R \times R^N \rightarrow R^N \quad and \quad \mathscr{Y}(t) = [\mathscr{Y}_1(t), \mathscr{Y}_2(t), \ldots, \mathscr{Y}_N(t)] .$$

Knowing that, f is vector of independent variables of N components of the system of ODE (1). To convert the function $f(t, y(t))$ which depends on two variables, to a function which depends only on one variable $\mathscr{Y}(t)$, using high dimension we can work in N+1 dimension using the assumption $\mathscr{Y}_{N+1}(t)$ t, then Equation (1) can be simplified to Equation (2):

$$v^{(7)}(t) = h(v(t)) \tag{2}$$

Using the following consideration:

$$\mathrm{v(t)} = \begin{pmatrix} \mathscr{Y}_1(\mathrm{t}) \\ \mathscr{Y}_2(\mathrm{t}) \\ \cdots \\ \cdots \\ \cdots \\ \mathscr{Y}_N(\mathrm{t}) \\ \mathrm{t}, \end{pmatrix}, \mathrm{h(v(t))} = \begin{pmatrix} f_1(v_1, v_2, \cdots, v_{N+1}) \\ f_2(v_1, v_2, \cdots, v_{N+1}) \\ f_3(v_1, v_2, \cdots, v_{N+1}) \\ \cdots \\ \cdots \\ \cdots \\ f_N(v_1, v_2, \cdots, v_{N+1}) \\ 0 \end{pmatrix}$$

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Subject to the initial condition $\mathscr{Y}'(t_0) = \overline{a_{\imath}}; = 0, 1, \ldots, 6$ where $\overline{a_{\imath}} = [\,\overline{a_1}\,, \overline{a_2}\,, \overline{a_3}\,, \ldots, \overline{a_N}\,, t_0]$ Some of engineers and scientists used to solve Equations (1) or (2) by multistep method (Lmm). Mostly, they used to solve ODEs of higher-order by converting them to equivalent system of first-order ODEs and they solved using a classical RK method [14]. However, it would be more efficient if seventh-order ODE can be solved using proposed direct RKM method. In this paper, we are concerned with direct explicit RKM integrators for solving class of seventh-order ODEs. Using Taylor series expansion approach, we have obtained the order conditions of the proposed methods. Consequently, we have derived two RKM integrators based on these algebraic order conditions.

3. Proposed RKM method

The formula of proposed explicit RKM integrator with s-stage for solving seventh-order ODE (1) written as following:

$$\mathscr{Y}_{n+1} = \mathscr{Y}_n + h\mathscr{Y}'_n + \frac{h^2}{2}\mathscr{Y}''_n + \frac{h^3}{6}\mathscr{Y}'''_n + \frac{h^4}{24}\mathscr{Y}''''_n + \frac{h^5}{120}\mathscr{Y}'''''_n + \frac{h^6}{720}\mathscr{Y}''''''_n + h^7\sum_{i=1}^{\mathscr{S}} b_i k_i, \tag{3}$$

$$\mathscr{Y}'_{n+1} = \mathscr{Y}'_n + h\mathscr{Y}''_n + \frac{h^2}{2}\mathscr{Y}'''_n + \frac{h^3}{6}\mathscr{Y}''''_n + \frac{h^4}{24}\mathscr{Y}'''''_n + \frac{h^5}{120}\mathscr{Y}''''''_n + h^6\sum_{i=1}^{\mathscr{S}} b'_i k_i, \tag{4}$$

$$\mathscr{Y}''_{n+1} = \mathscr{Y}''_n + h\mathscr{Y}'''_n + \frac{h^2}{2}\mathscr{Y}''''_n + \frac{h^3}{6}\mathscr{Y}'''''_n + \frac{h^4}{24}\mathscr{Y}''''''_n + h^5\sum_{i=1}^{\mathscr{S}} b''_i k_i, \tag{5}$$

$$\mathscr{Y}'''_{n+1} = \mathscr{Y}'''_n + h\mathscr{Y}''''_n + \frac{h^2}{2}\mathscr{Y}'''''_n + \frac{h^3}{6}\mathscr{Y}''''''_n + h^4\sum_{i=1}^{\mathscr{S}} b'''_i k_i, \tag{6}$$

$$\mathscr{Y}''''_{n+1} = \mathscr{Y}''''_n + h\mathscr{Y}'''''_n + \frac{h^2}{2}\mathscr{Y}''''''_n + h^3\sum_{i=1}^{\mathscr{S}} b''''_i k_i, \tag{7}$$

$$\mathscr{Y}'''''_{n+1} = \mathscr{Y}'''''_n + h\mathscr{Y}''''''_n + h^2\sum_{i=1}^{\mathscr{S}} b'''''_i k_i, \tag{8}$$

$$\mathscr{Y}''''''_{n+1}{}' = \mathscr{Y}''''''_n + h\sum_{i=1}^{\mathscr{S}} b''''''_i k_i, \tag{9}$$

$$k_1 = f(t_n\,, \mathscr{Y}_n), \tag{10}$$

and

$$k_i = f\left(t_n + c_i h,\ \mathscr{Y}_n + hc_i\mathscr{Y}'_n + \frac{h^2}{2}c_i^2\mathscr{Y}''_n + \frac{h^3}{6}c_i^3 y'''_n + \frac{h^4}{24}c_i^4\mathscr{Y}''''_n + \frac{h^5}{120}c_i^5\mathscr{Y}'''''_n + \frac{h^6}{720}c_i^6\mathscr{Y}''''''_n + h^7\sum_{j=1}^{i-1} a_{ij}k_j\right) \tag{11}$$

The parameters of RKM integrator are a_{ij}, c_i , b_i , b'_i , b''_i, b'''_i, b''''_i , b'''''_i and b''''''_i for $i, j = 1, 2, \ldots, \mathscr{S}$ are real and h is the step-size. RKM is an explicit integrator if $a_{ij} = 0$ for $i = j$ j and otherwise RKM is implicit integrator. The coefficients of RKM method have been expressed in Butcher table as in Table 1:

The order conditions of RKM integrators for solving fourth- and fifth-order ODEs have been derived by Refs. [12,13] respectively. In this study, using the same technique, we have derived the algebraic equations of order conditions of RKM methods for solving class of seventh-order ODEs.

3.1. The order conditions derivation of RKM methods

The order conditions of RKM integrators can be obtained from the direct expansion of the local truncation error. RKM formulae in (3–9) can be expressed as follows: $\mathscr{Y}^{(i)}_{n+1} = \mathscr{Y}_n + hw^{(i)}(t_n, \mathscr{Y}_n); i = 0.1, \ldots, 6$ where the increment functions are defined the following:

Table 1
Butcher table of RKM method.

C	A
	b^T
	b'^T
	b''^T
	b'''^T
	b''''^T
	b'''''^T
	b''''''^T

$$\omega(t_n,\mathscr{Y}_n)=\mathscr{Y}_n^{(1)}+\frac{h}{2}\mathscr{Y}''_n+\frac{h^2}{6}\mathscr{Y}'''_n+\frac{h^3}{24}\mathscr{Y}''''_n+\frac{h^4}{120}\mathscr{Y}'''''_n+\frac{h^5}{720}\mathscr{Y}''''''_n+h^6\sum_{i=1}^{\mathscr{S}}b_ik_i,$$

$$\omega^{(1)}(t_n,\mathscr{Y}_n)=\mathscr{Y}_n^{(2)}+\frac{h}{2}\mathscr{Y}'''_n+\frac{h^2}{6}\mathscr{Y}''''_n+\frac{h^3}{24}\mathscr{Y}'''''_n+\frac{h^4}{120}\mathscr{Y}''''''_n+h^5\sum_{i=1}^{\mathscr{S}}b'_ik'_i,$$

$$\omega^{(2)}(t_n,\mathscr{Y}_n)=\mathscr{Y}_n^{(3)}+\frac{h}{2}\mathscr{Y}''''_n+\frac{h^2}{6}\mathscr{Y}'''''_n+\frac{h^3}{24}\mathscr{Y}''''''_n+h^4\sum_{i=1}^{\mathscr{S}}b''_ik''_i,$$

$$\omega^{(3)}(t_n,\mathscr{Y}_n)=\mathscr{Y}_n^{(4)}+\frac{h}{2}\mathscr{Y}'''''_n+\frac{h^2}{6}\mathscr{Y}''''''_n+h^3\sum_{i=1}^{\mathscr{S}}b'''_ik'''_i,$$

$$\omega^{(4)}(t_n,\mathscr{Y}_n)=\mathscr{Y}_n^{(5)}+\frac{h}{2}\mathscr{Y}''''''_n+h^2\sum_{i=1}^{\mathscr{S}}b''''_ik''''_i,$$

$$\omega^{(5)}(t_n,\mathscr{Y}_n)=\mathscr{Y}_n^{(6)}+h\sum_{i=1}^{\mathscr{S}}b'''''_ik'''''_i,$$

$$\omega^{(6)}(t_n,\mathscr{Y}_n)=\sum_{i=1}^{\mathscr{S}}b''''''_ik''''''_i,$$

Where, k_i is defined as follow:

$$k_1=f(t_n,\ \mathscr{Y}_n) \tag{12}$$

$$k_i=f\left(t_n+c_ih,y_n+hc_iy'_n+\frac{h^2}{2}c_i^2y''_n+\frac{h^3}{6}c_i^3y'''_n+\frac{h^4}{24}c_i^4y''''_n+\frac{h^5}{120}c_i^5y'''''_n+\frac{h^6}{720}c_i^6y''''''_n+h^7\sum_{j=1}^{i-1}a_{ij}k_j,\right. \tag{13}$$

For i = 2,3,…,s

If Δ Taylor series increment function and the local truncation represents errors of the derivatives of the solution of order zero up to order seven can be obtained by substituting the analytical solution $\mathscr{Y}$ (t) of ODE (1) in to the RKM increment function . This gives

$$\mathscr{T}_{n+1}^{(i)}=h(\omega^{(i)}(t_n,\mathscr{Y}_n)-\Delta^{(i)}(t_n,\mathscr{Y}_n))$$

For i = 1, 2, 3, …,6. These expressions have given in elementary differentials terms also Taylor series increment can be expressed as follow :

$$\Delta^{(0)}=\mathscr{Y}'+\frac{h}{2}\mathscr{Y}''+\frac{h^2}{6}\mathscr{Y}'''+\frac{h^3}{24}\mathscr{Y}''''+\frac{h^4}{120}\mathscr{Y}'''''+\frac{h^5}{720}\mathscr{Y}''''''+O(h^6),$$

$$\Delta^{(1)}=\mathscr{Y}''+\frac{h}{2}\mathscr{Y}'''+\frac{h^2}{6}\mathscr{Y}''''+\frac{h^3}{24}\mathscr{Y}'''''+\frac{h^4}{120}\mathscr{Y}''''''+O(h^5),$$

$$\Delta^{(2)}=\mathscr{Y}'''+\frac{h}{2}\mathscr{Y}''''+\frac{h^2}{6}\mathscr{Y}'''''+\frac{h^3}{24}\mathscr{Y}''''''O(h^4),$$

$$\Delta^{(3)}=\mathscr{Y}''''+\frac{h}{2}\mathscr{Y}'''''+\frac{h^2}{6}\mathscr{Y}''''''+O(h^3),$$

$$\Delta^{(4)}=\mathscr{Y}'''''+\frac{h}{2}\mathscr{Y}''''''+O(h^3),$$

$$\Delta^{(5)}=\mathscr{Y}''''''+O(h\ ),$$

$$\Delta^{(6)}=O(1\ ),$$

Hence, for the scalar case function, the few first differentials of the function f are given as follow:

$$F_1^{(7)}=f\ ,$$

$$F_1^{(8)}=f_t+f_{\mathscr{Y}}\mathscr{Y}'\ ,$$

$$F_1^{(9)}=f_{tt}+2f_{t\mathscr{Y}}\mathscr{Y}'+f_{\mathscr{Y}}\mathscr{Y}''+f_{\mathscr{Y}\mathscr{Y}}(\mathscr{Y}')^2,$$

$$F_1^{(10)}=f_{ttt}+(f')^3f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}}+3(\mathscr{Y}')^2f_{t\mathscr{Y}\mathscr{Y}}+3\mathscr{Y}'\mathscr{Y}''f_{\mathscr{Y}\mathscr{Y}}+3\mathscr{Y}''f_{t\mathscr{Y}}+3\mathscr{Y}'f_{tt\mathscr{Y}}+\mathscr{Y}'''f_{\mathscr{Y}}\ ,$$

$$F_1^{(11)}=f_{tttt}+4\mathscr{Y}'f_{tt\mathscr{Y}}+6(\mathscr{Y}')^2f_{tt\mathscr{Y}\mathscr{Y}}+6\mathscr{Y}''f_{tt\mathscr{Y}}+4(\mathscr{Y}')^3f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}}+12\mathscr{Y}'\mathscr{Y}''f_{t\mathscr{Y}\mathscr{Y}}+4\mathscr{Y}'''f_{t\mathscr{Y}},$$

$$\begin{aligned}F_1^{(12)}&=f_{ttttt}+5\mathscr{Y}'f_{ttt\mathscr{Y}}+10(\mathscr{Y}')^2f_{ttt\mathscr{Y}\mathscr{Y}}+10\mathscr{Y}''f_{ttt\mathscr{Y}}+10(\mathscr{Y}')^3f_{tt\mathscr{Y}\mathscr{Y}\mathscr{Y}}+30\mathscr{Y}'\mathscr{Y}''f_{tt\mathscr{Y}\mathscr{Y}}\\&+10\mathscr{Y}'''f_{tt\mathscr{Y}}+30(\mathscr{Y}')^2\mathscr{Y}''f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}}+20\mathscr{Y}'\mathscr{Y}'''f_{t\mathscr{Y}\mathscr{Y}}+5(\mathscr{Y}')^4f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}+12(\mathscr{Y}'')^2f_{t\mathscr{Y}\mathscr{Y}}\\&+5\mathscr{Y}^{(4)}f_{t\mathscr{Y}}+10(\mathscr{Y}')^3\mathscr{Y}''f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}+6\mathscr{Y}'''f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}}+(\mathscr{Y}')^5f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}+\mathscr{Y}^{(5)}f_{\mathscr{Y}}\end{aligned}$$

$$F_1^{(13)}=f_{tttttt}+6\mathscr{Y}'f_{tttt\mathscr{Y}}+15(\mathscr{Y}')^2f_{ttt\mathscr{Y}\mathscr{Y}}+15\mathscr{Y}''f_{tttt\mathscr{Y}}+20(\mathscr{Y}')^3f_{tt\mathscr{Y}\mathscr{Y}\mathscr{Y}}$$

$$
\begin{aligned}
&+ 60\mathscr{Y}'\ \mathscr{Y}''f_{ttt\mathscr{Y}\mathscr{Y}} + 20\mathscr{Y}'''f_{ttt\mathscr{Y}} \\
&\quad + 15(\mathscr{Y}')^4\mathscr{Y}''f_{tt\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}} + 90\mathscr{Y}''(\mathscr{Y}')^2f_{tt\mathscr{Y}\mathscr{Y}\mathscr{Y}} \\
&+ 31\mathscr{Y}'\ \mathscr{Y}'''f_{tt\mathscr{Y}\mathscr{Y}} + 42(\mathscr{Y}'')^2f_{tt\mathscr{Y}\mathscr{Y}} + 15\mathscr{Y}^{(4)}f_{tt\mathscr{Y}} \\
&\quad + 60(\mathscr{Y}')^3\ \mathscr{Y}''f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}} \\
&+ 30(\mathscr{Y}')^2\mathscr{Y}'''f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}} + 60(\mathscr{Y}'')^2\ \mathscr{Y}'f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}} \\
&\quad + 20\mathscr{Y}'(\mathscr{Y}'')^2f_{tt\mathscr{Y}\mathscr{Y}} + 20(\mathscr{Y}')^2\mathscr{Y}''f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}} \\
&+ 30\mathscr{Y}'\ \mathscr{Y}^{(4)}f_{t\mathscr{Y}\mathscr{Y}} + 20\mathscr{Y}''\ \mathscr{Y}'''f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}} \\
&\quad + 5(\mathscr{Y}')^5f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}} + 12(\mathscr{Y}'')^2\ \mathscr{Y}'f_{t\mathscr{Y}\mathscr{Y}} \\
&\quad + 28\mathscr{Y}''\ \mathscr{Y}'''f_{t\mathscr{Y}\mathscr{Y}} \\
&+ 6\mathscr{Y}^{(5)}f_{t\mathscr{Y}} + 15(\mathscr{Y}')^4\mathscr{Y}''f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}} \\
&\quad + 10(\mathscr{Y}')^3\mathscr{Y}'''f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}} + 30(\mathscr{Y}')^2(\mathscr{Y}'')^2f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}} \\
&+ 10\mathscr{Y}'''(\mathscr{Y}')^2f_{tt\mathscr{Y}\mathscr{Y}\mathscr{Y}} + 10(\mathscr{Y}')^3\mathscr{Y}'''f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}} \\
&\quad + 15(\mathscr{Y}')^2\ \mathscr{Y}^{(4)}f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}} + 20\mathscr{Y}'\ \mathscr{Y}''\ \mathscr{Y}'''f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}} \\
&+ 9\mathscr{Y}^{(4)})\mathscr{Y}''f_{\mathscr{Y}\mathscr{Y}} + 6\mathscr{Y}'\ \mathscr{Y}^{(5)}f_{\mathscr{Y}\mathscr{Y}} \\
&\quad + 12(\mathscr{Y}'')^3f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}} + 4\mathscr{Y}'\mathscr{Y}''\mathscr{Y}'''f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}} \\
&\quad + 4(\mathscr{Y}''')^2f_{\mathscr{Y}\mathscr{Y}} \\
&+ 3\mathscr{Y}''f_{tt\mathscr{Y}\mathscr{Y}} + 3\mathscr{Y}''\ \mathscr{Y}'''f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}} + 5(\mathscr{Y}')^5f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}} \\
&\quad + 12(\mathscr{Y}'')^2\ \mathscr{Y}'f_{t\mathscr{Y}\mathscr{Y}} + 28\mathscr{Y}''\ \mathscr{Y}'''f_{t\mathscr{Y}\mathscr{Y}} \\
&+ 6\mathscr{Y}^{(5)}f_{x\mathscr{Y}} + 15(\mathscr{Y}')^4\ \mathscr{Y}''f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}} \\
&\quad + 10(\mathscr{Y}')^3\ \mathscr{Y}'\ \mathscr{Y}''f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}} + 3\mathscr{Y}'''f_{t\mathscr{Y}\mathscr{Y}} \\
&\quad + 3\mathscr{Y}''\ \mathscr{Y}''\mathscr{Y}'f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}} \\
&+ 3(\mathscr{Y}')^2\mathscr{Y}''f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}} + 9\mathscr{Y}'\mathscr{Y}'''f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}} \\
&\quad + 3(\mathscr{Y}'')f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}} + 6\mathscr{Y}'''f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}} + 6\mathscr{Y}^{(4)}f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}} \\
&+ (\mathscr{Y}')^5f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}} + (\mathscr{Y}')^6f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}} + \mathscr{Y}^{(6)}f_{\mathscr{Y}}\ .
\end{aligned}
$$

The increment functions $\omega^{(i)}\ ; i = 0, 1, \ldots, 6$ for the RKM integrators can be written using the above terms:

$$
\begin{aligned}
\sum_{i=1}^{\mathscr{S}} b_ik_i &= \sum_{i=1}^{\mathscr{S}} b_if + \sum_{i=1}^{\mathscr{S}} b_ic_i(f_t + f_{\mathscr{Y}}\ \mathscr{Y}')h \\
&+ \frac{1}{2}\sum_{i=1}^{\mathscr{S}} b_ic_i^2\big(f_{tt} + 2f_{t\mathscr{Y}}\mathscr{Y}' + f_{\mathscr{Y}}\mathscr{Y}'' + f_{\mathscr{Y}\mathscr{Y}}(\mathscr{Y}')^2\big)h^2 \\
&+ \frac{1}{6}\sum_{i=1}^{\mathscr{S}} b_ic_i^3\big(f_{ttt} + (f')^3f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}} + 3\mathscr{Y}'\ \mathscr{Y}''f_{\mathscr{Y}\mathscr{Y}} \\
&\quad + 3\mathscr{Y}''f_{t\mathscr{Y}} + 3\mathscr{Y}'f_{tt\mathscr{Y}} + \mathscr{Y}'''f_{\mathscr{Y}}\big)h^3 \\
&+ \frac{1}{24}\sum_{i=1}^{\mathscr{S}} b_ic_i^4\Big(f_{tttt} + 4\mathscr{Y}'f_{ttt\mathscr{Y}} + 6(\mathscr{Y}')^2f_{tt\mathscr{Y}\mathscr{Y}} \\
&\quad + 6\mathscr{Y}''f_{tt\mathscr{Y}} + 4(\mathscr{Y}')^3f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}} + 12\mathscr{Y}'\ \mathscr{Y}''f_{t\mathscr{Y}\mathscr{Y}} \\
&\quad + 4\mathscr{Y}'''f_{t\mathscr{Y}} \\
&+ 6(\mathscr{Y}')^2\mathscr{Y}''f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}} + 4\mathscr{Y}'\ \mathscr{Y}'''f_{\mathscr{Y}\mathscr{Y}} + 3\mathscr{Y}''f_{\mathscr{Y}\mathscr{Y}} \\
&\quad + (\mathscr{Y}')^4f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}} + y^{(4)}f_{\mathscr{Y}}\Big)h^4 \\
&+ \frac{1}{120}\sum_{i=1}^{\mathscr{S}} b_ic_i^5\Big(f_{ttttt} + 5\mathscr{Y}'f_{tttt\mathscr{Y}} + 10(\mathscr{Y}')^2f_{ttt\mathscr{Y}\mathscr{Y}} \\
&\quad + 10\mathscr{Y}''f_{ttt\mathscr{Y}} + 10(\mathscr{Y}')^3f_{tt\mathscr{Y}\mathscr{Y}\mathscr{Y}} + 30\mathscr{Y}'\ \mathscr{Y}''f_{tt\mathscr{Y}\mathscr{Y}} \\
&+ 10\ \mathscr{Y}'''f_{tt\mathscr{Y}} + 30(\mathscr{Y}')^2\mathscr{Y}''f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}} \\
&\quad + 20\mathscr{Y}'\ \mathscr{Y}'''f_{t\mathscr{Y}\mathscr{Y}} \\
&\quad + 5(\mathscr{Y}')^4f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}} + 12(\mathscr{Y}'')^2f_{t\mathscr{Y}\mathscr{Y}} + 5\mathscr{Y}^{(4)}f_{t\mathscr{Y}} \\
&\quad + 10(\mathscr{Y}')^3\ \mathscr{Y}''f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}} + 10(\mathscr{Y}')^2\mathscr{Y}'''f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}} \\
&\quad + 5\mathscr{Y}^{(4)}\mathscr{Y}' + f_{\mathscr{Y}\mathscr{Y}} + 12\mathscr{Y}'(\mathscr{Y}'')^2f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}} \\
&\quad + 4\mathscr{Y}''\ \mathscr{Y}'''f_{\mathscr{Y}\mathscr{Y}} + 3\mathscr{Y}''f_{t\mathscr{Y}\mathscr{Y}} \\
&\quad + 3\mathscr{Y}'\ \mathscr{Y}''f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}} + 6\mathscr{Y}'''f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}} + (\mathscr{Y}')^5f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}} \\
&\quad + \mathscr{Y}^{(5)}f_{\mathscr{Y}}\Big)h^5 \\
&+ \frac{1}{720}\sum_{i=1}^{\mathscr{S}} b_ic_i^6\Big(f_{tttttt} + 6\mathscr{Y}'f_{ttttt\mathscr{Y}} + 15(\mathscr{Y}')^2f_{tttt\mathscr{Y}\mathscr{Y}} \\
&\quad + 15\mathscr{Y}''f_{tttt\mathscr{Y}} + 20(\mathscr{Y}')^3f_{ttt\mathscr{Y}\mathscr{Y}\mathscr{Y}} + 60\mathscr{Y}'\ \mathscr{Y}''f_{ttt\mathscr{Y}\mathscr{Y}} \\
&\quad + 20\mathscr{Y}'''f_{ttt\mathscr{Y}} + 15(\mathscr{Y}')^4f_{tt\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}} \\
&\quad + 90\mathscr{Y}''(\mathscr{Y}')^2f_{tt\mathscr{Y}\mathscr{Y}\mathscr{Y}} + 31\mathscr{Y}'\mathscr{Y}'''f_{tt\mathscr{Y}\mathscr{Y}\mathscr{Y}} \\
&\quad + 42(\mathscr{Y}')^2f_{tt\mathscr{Y}\mathscr{Y}} + 15\mathscr{Y}^{(4)}f_{tt\mathscr{Y}} \\
&+ 60(\mathscr{Y}')^3\mathscr{Y}''f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}} + 30(\mathscr{Y}')^2\mathscr{Y}'''f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}} \\
&\quad + 60(\mathscr{Y}'')^2\mathscr{Y}'f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}} + 20\mathscr{Y}'(\mathscr{Y}'')^2f_{tt\mathscr{Y}\mathscr{Y}} \\
&\quad + 20\mathscr{Y}'(\mathscr{Y}'')^2\mathscr{Y}''f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}} \\
&\quad + 30\mathscr{Y}'\mathscr{Y}^{(4)}f_{t\mathscr{Y}\mathscr{Y}} + 20\mathscr{Y}''\mathscr{Y}'''f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}} \\
&\quad + 5(\mathscr{Y}')^5f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}} + 12(\mathscr{Y}'')^2\mathscr{Y}'f_{t\mathscr{Y}\mathscr{Y}} \\
&\quad + 28\mathscr{Y}''\ \mathscr{Y}'''f_{t\mathscr{Y}\mathscr{Y}} + 6\mathscr{Y}^{(5)}f_{t\mathscr{Y}}
\end{aligned}
$$

$$+15(\mathscr{Y}')^4\mathscr{Y}''f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}+10(\mathscr{Y}')^3\mathscr{Y}'''f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}$$
$$+30(\mathscr{Y}')^2(\mathscr{Y}'')^2f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}+10\mathscr{Y}'''(\mathscr{Y}')^2f_{tt\mathscr{Y}\mathscr{Y}\mathscr{Y}}$$
$$+10(\mathscr{Y}')^3\mathscr{Y}'''f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}$$

$$+15(\mathscr{Y}')^2\mathscr{Y}^{(4)}f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}}+20\mathscr{Y}'\ \mathscr{Y}''\ \mathscr{Y}'''f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}}$$
$$+9\mathscr{Y}^{(4)})\mathscr{Y}''f_{\mathscr{Y}\mathscr{Y}}+6\mathscr{Y}'\mathscr{Y}^{(5)}f_{\mathscr{Y}\mathscr{Y}}$$
$$+12(\mathscr{Y}'')^3f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}}+4\mathscr{Y}'\mathscr{Y}''\mathscr{Y}'''f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}$$

$$+4(\mathscr{Y}''')^2f_{\mathscr{Y}\mathscr{Y}}+3\mathscr{Y}''f_{tt\mathscr{Y}\mathscr{Y}}+3\mathscr{Y}''\mathscr{Y}'''f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}}$$
$$+5(\mathscr{Y}')^5f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}+12(\mathscr{Y}'')^2\mathscr{Y}'f_{t\mathscr{Y}\mathscr{Y}}$$
$$+28\mathscr{Y}''\mathscr{Y}'''f_{t\mathscr{Y}\mathscr{Y}}+6\mathscr{Y}^{(5)}f_{t\mathscr{Y}}$$

$$+15(\mathscr{Y}')^4\mathscr{Y}''f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}+10(\mathscr{Y}')3\ \mathscr{Y}'\mathscr{Y}''f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}}$$
$$+3\mathscr{Y}'''f_{t\mathscr{Y}\mathscr{Y}}+3\mathscr{Y}''\mathscr{Y}'f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}}$$
$$+3(\mathscr{Y}')^2\mathscr{Y}''f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}+9\mathscr{Y}'\mathscr{Y}'''f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}}$$

$$+3(\mathscr{Y}'')f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}}+6\mathscr{Y}'''f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}}+6\mathscr{Y}^{(4)}f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}}$$
$$+(\mathscr{Y}')^5f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}+(\mathscr{Y}')^6f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}$$
$$+\mathscr{Y}^{(6)}f_{\mathscr{Y}}\Big)h^6+O\big(h^7\big) \qquad (14)$$

$$\sum_{i=1}^{\mathscr{S}}b_i'k_i=\sum_{i=1}^{\mathscr{S}}b_i'f+\sum_{i=1}^{\mathscr{S}}b_i'c_i(f_t+f_{\mathscr{Y}}\mathscr{Y}')h$$
$$+\frac{1}{2}\sum_{i=1}^{\mathscr{S}}b_i'c_i^2\big(f_{tt}+2f_{t\mathscr{Y}}\mathscr{Y}'+f_{\mathscr{Y}}\mathscr{Y}''+f_{\mathscr{Y}\mathscr{Y}}(\mathscr{Y}')^2\big)h^2$$

$$+\frac{1}{6}\sum_{i=1}^{\mathscr{S}}b_i'c_i^3\big(f_{ttt}+(f')^3f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}}+3(\mathscr{Y}')^2f_{t\mathscr{Y}\mathscr{Y}}$$
$$+3\mathscr{Y}'\mathscr{Y}''f_{\mathscr{Y}\mathscr{Y}}+3\mathscr{Y}''f_{t\mathscr{Y}}+3\mathscr{Y}'f_{tt\mathscr{Y}}+\mathscr{Y}'''f_{\mathscr{Y}}\big)h^3$$

$$+\frac{1}{24}\sum_{i=1}^{\mathscr{S}}b_i'c_i^4\Big(f_{ttt}+4\mathscr{Y}'f_{ttt\mathscr{Y}}+6(\mathscr{Y}')^2f_{tt\mathscr{Y}\mathscr{Y}}$$
$$+6\mathscr{Y}''f_{tt\mathscr{Y}}+4(\mathscr{Y}')^3f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}}+12\mathscr{Y}'\mathscr{Y}''f_{t\mathscr{Y}\mathscr{Y}}$$
$$+4\mathscr{Y}'''f_{t\mathscr{Y}}$$
$$+6(\mathscr{Y}')^2\mathscr{Y}''f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}}+4\mathscr{Y}'\mathscr{Y}'''f_{\mathscr{Y}\mathscr{Y}}$$
$$+3\mathscr{Y}''f_{\mathscr{Y}\mathscr{Y}}+(\mathscr{Y}')^4f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}+\mathscr{Y}^{(4)}f_{\mathscr{Y}}\Big)h^4$$

$$+\frac{1}{120}\sum_{i=1}^{\mathscr{S}}b_i'c_i^5\Big(f_{ttttt}+5\mathscr{Y}'f_{tttt\mathscr{Y}}+10(\mathscr{Y}')^2f_{ttt\mathscr{Y}\mathscr{Y}}$$
$$+10\mathscr{Y}''f_{ttt\mathscr{Y}}+10(\mathscr{Y}')^3f_{tt\mathscr{Y}\mathscr{Y}\mathscr{Y}}+30\mathscr{Y}'\mathscr{Y}''f_{tt\mathscr{Y}\mathscr{Y}}$$

$$+10\mathscr{Y}'''f_{tt\mathscr{Y}}+30(\mathscr{Y}')^2\mathscr{Y}''f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}}$$
$$+20\mathscr{Y}'\mathscr{Y}'''f_{t\mathscr{Y}\mathscr{Y}}+5(\mathscr{Y}')^4f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}$$
$$+12(\mathscr{Y}'')^2f_{t\mathscr{Y}\mathscr{Y}}+5\mathscr{Y}^{(4)}f_{t\mathscr{Y}}$$

$$+10(\mathscr{Y}')^3\mathscr{Y}''f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}+10(\mathscr{Y}')^2\mathscr{Y}'''f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}}$$
$$+5\mathscr{Y}^{(4)}\mathscr{Y}'f_{\mathscr{Y}\mathscr{Y}}+12\mathscr{Y}'(\mathscr{Y}')^2f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}}$$
$$+4\mathscr{Y}''\mathscr{Y}'''f_{\mathscr{Y}\mathscr{Y}}+3\mathscr{Y}''f_{t\mathscr{Y}\mathscr{Y}}$$

$$+3\mathscr{Y}'\ \mathscr{Y}''f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}}+6\mathscr{Y}'''f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}}$$
$$+(\mathscr{Y}')^5f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}+\mathscr{Y}^{(5)}f_{\mathscr{Y}}\Big)h^5$$

$$+\frac{1}{720}\sum_{i=1}^{\mathscr{S}}b_i'c_i^6\Big(f_{tttttt}+6\mathscr{Y}'f_{ttttt\mathscr{Y}}+15(\mathscr{Y}')^2f_{tttt\mathscr{Y}\mathscr{Y}}$$
$$+15\mathscr{Y}''f_{tttt\mathscr{Y}}+20(\mathscr{Y}')^3f_{ttt\mathscr{Y}\mathscr{Y}\mathscr{Y}}+60\mathscr{Y}'\mathscr{Y}''f_{ttt\mathscr{Y}\mathscr{Y}}$$

$$+20\mathscr{Y}'''f_{ttt\mathscr{Y}}+15(\mathscr{Y}')^4\mathscr{Y}''f_{tt\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}$$
$$+90\mathscr{Y}''(\mathscr{Y}')^2f_{tt\mathscr{Y}\mathscr{Y}\mathscr{Y}}+31\mathscr{Y}'\mathscr{Y}'''f_{tt\mathscr{Y}\mathscr{Y}}$$
$$+42(\mathscr{Y}'')^2f_{tt\mathscr{Y}\mathscr{Y}}+15\mathscr{Y}^{(4)}f_{tt\mathscr{Y}}$$

$$+60(\mathscr{Y}')^3\mathscr{Y}''f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}+30(\mathscr{Y}')^2\mathscr{Y}'''f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}}$$
$$+60(\mathscr{Y}'')^2\mathscr{Y}'f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}}+20\mathscr{Y}'(\mathscr{Y}'')^2f_{tt\mathscr{Y}\mathscr{Y}}$$
$$+20(\mathscr{Y}')^2\mathscr{Y}''f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}}$$

$$+30\mathscr{Y}'\mathscr{Y}^{(4)}f_{t\mathscr{Y}\mathscr{Y}}+20\mathscr{Y}''\mathscr{Y}'''f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}}$$
$$+5(\mathscr{Y}')^5f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}+12(\mathscr{Y}'')^2\mathscr{Y}'f+t\mathscr{Y}\mathscr{Y}$$
$$+28\mathscr{Y}''\mathscr{Y}'''f_{t\mathscr{Y}\mathscr{Y}}+6\mathscr{Y}^{(5)}f_{t\mathscr{Y}}$$

$$+15(\mathscr{Y}')^4\mathscr{Y}''f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}+10(\mathscr{Y}')^3\mathscr{Y}'''f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}$$
$$+30(\mathscr{Y}')^2(\mathscr{Y}'')^2f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}$$
$$+10\mathscr{Y}'''(\mathscr{Y}')^2f_{tt\mathscr{Y}\mathscr{Y}\mathscr{Y}}$$
$$+10(\mathscr{Y}')^3\mathscr{Y}'''f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}$$
$$+15(\mathscr{Y}')^2\mathscr{Y}^{(4)}f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}}+20\mathscr{Y}'\mathscr{Y}''\mathscr{Y}'''f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}}$$
$$+9\mathscr{Y}^{(4)}\mathscr{Y}'''f_{\mathscr{Y}\mathscr{Y}}+6\mathscr{Y}'\ \mathscr{Y}^{(5)}f_{\mathscr{Y}\mathscr{Y}}$$
$$+12(\mathscr{Y}'')^3f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}}+4\mathscr{Y}'\mathscr{Y}''\mathscr{Y}'''f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}$$

$$+4(\mathscr{Y}''')^2f_{\mathscr{Y}\mathscr{Y}}+3\mathscr{Y}''f_{tt\mathscr{Y}\mathscr{Y}}+3\mathscr{Y}''\mathscr{Y}'''f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}}$$
$$+5(\mathscr{Y}')^5f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}+12(\mathscr{Y}'')^2\mathscr{Y}'f_{t\mathscr{Y}\mathscr{Y}}$$
$$+28\mathscr{Y}''\mathscr{Y}'''f_{t\mathscr{Y}\mathscr{Y}}+6\mathscr{Y}^{(5)}f_{t\mathscr{Y}}$$

$$
\begin{aligned}
&+15(\mathscr{Y}')^4\mathscr{Y}''f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}\\
&\quad+10(\mathscr{Y}')^3\mathscr{Y}'\mathscr{Y}''f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}}\\
&\quad+3\mathscr{Y}'''f_{t\mathscr{Y}\mathscr{Y}}+3\mathscr{Y}''\mathscr{Y}'f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}}\\
&\quad+3(\mathscr{Y}')^2\mathscr{Y}''f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}+9\mathscr{Y}'\mathscr{Y}'''f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}}\\
&+3(\mathscr{Y}'')f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}}+6\mathscr{Y}'''f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}}+6\mathscr{Y}^{(4)}f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}}\\
&\quad+(\mathscr{Y}')^5f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}+(\mathscr{Y}')^6f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}\\
&\quad+\mathscr{Y}^{(6)}f_{\mathscr{Y}}\Big)h^6+O(h^7)
\end{aligned}
\tag{15}
$$

$$
\begin{aligned}
\sum_{i=1}^{\mathscr{S}}b_i''k_i&=\sum_{i=1}^{\mathscr{S}}b_i''f+\sum_{i=1}^{\mathscr{S}}b_i''c_i(f_t+f_{\mathscr{Y}}\mathscr{Y}')h\\
&+\frac{1}{2}\sum_{i=1}^{\mathscr{S}}b_i''c_i^2\left(f_{tt}+2f_{t\mathscr{Y}}\mathscr{Y}'+f_{\mathscr{Y}}\mathscr{Y}''+f_{\mathscr{Y}\mathscr{Y}}(\mathscr{Y}')^2\right)h^2\\
&+\frac{1}{6}\sum_{i=1}^{\mathscr{S}}b_i''c_i^3\big(f_{ttt}+(f')^3f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}}+3(\mathscr{Y}')^2f_{t\mathscr{Y}\mathscr{Y}}\\
&\quad+3\mathscr{Y}'\mathscr{Y}''f_{\mathscr{Y}\mathscr{Y}}+3\mathscr{Y}''f_{t\mathscr{Y}}+3\mathscr{Y}'f_{tt\mathscr{Y}}\\
&\quad+\mathscr{Y}'''f_{\mathscr{Y}}\big)h^3\\
&+\frac{1}{24}\sum_{i=1}^{\mathscr{S}}b_i''c_i^4\Big(f_{tttt}+4\mathscr{Y}'f_{ttt\mathscr{Y}}+6(\mathscr{Y}')^2f_{tt\mathscr{Y}\mathscr{Y}}\\
&\quad+6\mathscr{Y}''f_{tt\mathscr{Y}}+4(\mathscr{Y}')^3f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}}\\
&\quad+12\mathscr{Y}'\mathscr{Y}''f_{t\mathscr{Y}\mathscr{Y}}+4\mathscr{Y}'''f_{t\mathscr{Y}}\\
&+6(\mathscr{Y}')^2\mathscr{Y}''f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}}+4\mathscr{Y}'\mathscr{Y}'''f_{\mathscr{Y}\mathscr{Y}}\\
&\quad+3\mathscr{Y}''f_{\mathscr{Y}\mathscr{Y}}+(\mathscr{Y}')^4f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}+\mathscr{Y}^{(4)}f_{\mathscr{Y}}\Big)h^4\\
&+\frac{1}{120}\sum_{i=1}^{\mathscr{S}}b_i''c_i^5\Big(f_{ttttt}+5\mathscr{Y}'f_{tttt\mathscr{Y}}\\
&\quad+10(\mathscr{Y}')^2f_{ttt\mathscr{Y}\mathscr{Y}}+10\mathscr{Y}''f_{ttt\mathscr{Y}}\\
&\quad+10(\mathscr{Y}')^3f_{tt\mathscr{Y}\mathscr{Y}\mathscr{Y}}+30\mathscr{Y}'\mathscr{Y}''f_{tt\mathscr{Y}\mathscr{Y}}\\
&+10\mathscr{Y}'''f_{tt\mathscr{Y}}+30(\mathscr{Y}')^2\mathscr{Y}''f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}}\\
&\quad+20\mathscr{Y}'\mathscr{Y}'''f_{t\mathscr{Y}\mathscr{Y}}+5(\mathscr{Y}')^4f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}\\
&\quad+12(\mathscr{Y}'')^2f_{t\mathscr{Y}\mathscr{Y}}+5\mathscr{Y}^{(4)}f_{t\mathscr{Y}}\\
&+10(\mathscr{Y}')^3\mathscr{Y}''f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}+10(\mathscr{Y}')^2\mathscr{Y}'''f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}}\\
&\quad+5\mathscr{Y}^{(4)}\mathscr{Y}'f_{\mathscr{Y}\mathscr{Y}}+12\mathscr{Y}'(\mathscr{Y}'')^2f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}}\\
&\quad+4\mathscr{Y}''\mathscr{Y}'''f_{\mathscr{Y}\mathscr{Y}}+3\mathscr{Y}''f_{t\mathscr{Y}\mathscr{Y}}\\
&\quad+3\mathscr{Y}'\mathscr{Y}''f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}}+6\mathscr{Y}'''f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}}\\
&\quad+(\mathscr{Y}')^5f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}+\mathscr{Y}^{(5)}f_{\mathscr{Y}}\Big)h^5\\
&+\frac{1}{720}\sum_{i=1}^{\mathscr{S}}b_i''c_i^6\Big(f_{tttttt}+6\mathscr{Y}'f_{ttttt\mathscr{Y}}\\
&\quad+15(\mathscr{Y}')^2f_{tttt\mathscr{Y}\mathscr{Y}}+15\mathscr{Y}''f_{tttt\mathscr{Y}}\\
&\quad+20(\mathscr{Y}')^3f_{ttt\mathscr{Y}\mathscr{Y}\mathscr{Y}}+60\mathscr{Y}'\mathscr{Y}''f_{ttt\mathscr{Y}\mathscr{Y}}\\
&+20\mathscr{Y}''f_{ttt\mathscr{Y}}+15(\mathscr{Y}')^4\mathscr{Y}''f_{tt\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}\\
&\quad+90\mathscr{Y}''(\mathscr{Y}')^2f_{tt\mathscr{Y}\mathscr{Y}\mathscr{Y}}+31\mathscr{Y}'\mathscr{Y}'''f_{tt\mathscr{Y}\mathscr{Y}}\\
&\quad+42(\mathscr{Y}'')^2f_{tt\mathscr{Y}\mathscr{Y}}+15\mathscr{Y}^{(4)}f_{tt\mathscr{Y}}\\
&+60(\mathscr{Y}')^3\mathscr{Y}''f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}+30(\mathscr{Y}')^2\mathscr{Y}'''f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}}\\
&\quad+60(\mathscr{Y}'')^2\mathscr{Y}'f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}}+20\mathscr{Y}'(\mathscr{Y}'')^2f_{tt\mathscr{Y}\mathscr{Y}}\\
&\quad+20(\mathscr{Y}'')^2\mathscr{Y}''f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}}\\
&+30\mathscr{Y}'\mathscr{Y}^{(4)}f_{t\mathscr{Y}\mathscr{Y}}+20\mathscr{Y}''\mathscr{Y}'''f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}}\\
&\quad+5(\mathscr{Y}')^5f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}+12(\mathscr{Y}'')^2\mathscr{Y}'f_{t\mathscr{Y}\mathscr{Y}}\\
&\quad+28\mathscr{Y}''\mathscr{Y}'''f_{t\mathscr{Y}\mathscr{Y}}+6\mathscr{Y}^{(5)}f_{t\mathscr{Y}}\\
&+15(\mathscr{Y}')^4\mathscr{Y}''f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}+10(\mathscr{Y}')^3\mathscr{Y}'''f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}\\
&\quad+30(\mathscr{Y}')^2(\mathscr{Y}'')^2f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}\\
&\quad+10\mathscr{Y}'''(\mathscr{Y}')^2f_{tt\mathscr{Y}\mathscr{Y}}\\
&\quad+10(\mathscr{Y}')^3\mathscr{Y}'''f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}\\
&+15(\mathscr{Y}')^2\mathscr{Y}^{(4)}f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}}+20\mathscr{Y}'\mathscr{Y}''\mathscr{Y}'''f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}}\\
&\quad+9\mathscr{Y}^{(4)})\mathscr{Y}''f_{\mathscr{Y}\mathscr{Y}}+6\mathscr{Y}'\mathscr{Y}^{(5)}f_{\mathscr{Y}\mathscr{Y}}\\
&\quad+12(\mathscr{Y}'')^3f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}}+4\mathscr{Y}'\mathscr{Y}''\mathscr{Y}'''f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}\\
&+4(\mathscr{Y}''')^2f_{\mathscr{Y}\mathscr{Y}}+3\mathscr{Y}''f_{tt\mathscr{Y}\mathscr{Y}}+3\mathscr{Y}''\mathscr{Y}'''f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}}\\
&\quad+5(\mathscr{Y}')^5f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}+12(\mathscr{Y}'')^2\mathscr{Y}'f_{t\mathscr{Y}\mathscr{Y}}\\
&\quad+28\mathscr{Y}''\mathscr{Y}'''f_{t\mathscr{Y}\mathscr{Y}}+6\mathscr{Y}^{(5)}f_{t\mathscr{Y}}\\
&+15(\mathscr{Y}')^4\mathscr{Y}''f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}\\
&\quad+10(\mathscr{Y}')^3\mathscr{Y}'\mathscr{Y}''f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}}+3\mathscr{Y}'''\\
&\quad+3\mathscr{Y}''\mathscr{Y}'f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}}+3(\mathscr{Y}')^2\mathscr{Y}''f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}\\
&\quad+9\mathscr{Y}'\mathscr{Y}'''f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}}
\end{aligned}
$$

$$+3(\mathscr{Y}'')f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}}+6\mathscr{Y}'''f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}}+6\mathscr{Y}^{(4)}f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}}
+(\mathscr{Y}')^5f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}+(\mathscr{Y}')^6f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}
+\mathscr{Y}^{(6)}f_{\mathscr{Y}}\Big)h^6+O\left(h^7\right) \tag{16}$$

$$\sum_{i=1}^{\mathscr{S}}b_i'''k_i=\sum_{i=1}^{\mathscr{S}}b_i'''f+\sum_{i=1}^{\mathscr{S}}b_i'''c_i(f_t+f_{\mathscr{Y}}\mathscr{Y}')h
+\frac{1}{2}\sum_{i=1}^{\mathscr{S}}b_i'''c_i^2\left(f_{tt}+2f_{t\mathscr{Y}}\mathscr{Y}'+f_{\mathscr{Y}}\mathscr{Y}''+f_{\mathscr{Y}\mathscr{Y}}(\mathscr{Y}')^2\right)h^2$$

$$+\frac{1}{6}\sum_{i=1}^{\mathscr{S}}b_i'''c_i^3\big(f_{ttt}+(f')^3f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}}+3(\mathscr{Y}')^2f_{t\mathscr{Y}\mathscr{Y}}
+3\mathscr{Y}'\mathscr{Y}''f_{\mathscr{Y}\mathscr{Y}}+3\mathscr{Y}''f_{t\mathscr{Y}}+3\mathscr{Y}'f_{tt\mathscr{Y}}
+\mathscr{Y}'''f_{\mathscr{Y}}\big)h^3$$

$$+\frac{1}{24}\sum_{i=1}^{\mathscr{S}}b_i'''c_i^4\Big(f_{tttt}+4\mathscr{Y}'f_{ttt\mathscr{Y}}+6(\mathscr{Y}')^2f_{tt\mathscr{Y}\mathscr{Y}}
+6\mathscr{Y}''f_{tt\mathscr{Y}}+4(\mathscr{Y}')^3f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}}
+12\mathscr{Y}'\mathscr{Y}''f_{t\mathscr{Y}\mathscr{Y}}+4\mathscr{Y}'''f_{t\mathscr{Y}}$$

$$+6(\mathscr{Y}')^2\mathscr{Y}''f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}}+4\mathscr{Y}'\mathscr{Y}'''f_{\mathscr{Y}\mathscr{Y}}
+3\mathscr{Y}''f_{\mathscr{Y}\mathscr{Y}}+(\mathscr{Y}')^4f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}+\mathscr{Y}^{(4)}f_{\mathscr{Y}}\Big)h^4$$

$$+\frac{1}{120}\sum_{i=1}^{\mathscr{S}}b_i'''c_i^5\Big(f_{ttttt}+5\mathscr{Y}'f_{tttt\mathscr{Y}}
+10(\mathscr{Y}')^2f_{ttt\mathscr{Y}\mathscr{Y}}+10\mathscr{Y}''f_{ttt\mathscr{Y}}
+10(\mathscr{Y}')^3f_{tt\mathscr{Y}\mathscr{Y}\mathscr{Y}}+30\mathscr{Y}'\mathscr{Y}''f_{tt\mathscr{Y}\mathscr{Y}}$$

$$+10\mathscr{Y}'''f_{tt\mathscr{Y}}+30(\mathscr{Y}')^2\mathscr{Y}''f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}}
+20\mathscr{Y}'\mathscr{Y}'''f_{t\mathscr{Y}\mathscr{Y}}+5(\mathscr{Y}')^4f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}
+12(\mathscr{Y}'')^2f_{t\mathscr{Y}\mathscr{Y}}+5\mathscr{Y}^{(4)}f_{t\mathscr{Y}}$$

$$+10(\mathscr{Y}')^3\mathscr{Y}''f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}+10(\mathscr{Y}')^2\mathscr{Y}'''f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}}
+5\mathscr{Y}^{(4)}\mathscr{Y}'f_{\mathscr{Y}\mathscr{Y}}+12\mathscr{Y}'(\mathscr{Y}'')^2f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}}
+4\mathscr{Y}''\mathscr{Y}'''f_{\mathscr{Y}\mathscr{Y}}+3\mathscr{Y}''f_{t\mathscr{Y}\mathscr{Y}}$$

$$+3\mathscr{Y}'\mathscr{Y}''f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}}+6\mathscr{Y}'''f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}}
+(\mathscr{Y}')^5f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}+\mathscr{Y}^{(5)}f_{\mathscr{Y}}\Big)h^5$$

$$+\frac{1}{720}\sum_{i=1}^{\mathscr{S}}b_i'''c_i^6\Big(f_{tttttt}+6\mathscr{Y}'f_{ttttt\mathscr{Y}}
+15(\mathscr{Y}')^2f_{tttt\mathscr{Y}\mathscr{Y}}+15\mathscr{Y}''f_{tttt\mathscr{Y}}
+20(\mathscr{Y}')^3f_{ttt\mathscr{Y}\mathscr{Y}\mathscr{Y}}+60\mathscr{Y}'\mathscr{Y}''f_{ttt\mathscr{Y}\mathscr{Y}}$$

$$+20\mathscr{Y}'''f_{ttt\mathscr{Y}}+15(\mathscr{Y}')^4\mathscr{Y}''f_{tt\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}
+90\mathscr{Y}''(\mathscr{Y}')^2f_{tt\mathscr{Y}\mathscr{Y}\mathscr{Y}}+31\mathscr{Y}'\mathscr{Y}'''f_{tt\mathscr{Y}\mathscr{Y}}
+42(\mathscr{Y}'')^2f_{tt\mathscr{Y}\mathscr{Y}}+15\mathscr{Y}^{(4)}f_{tt\mathscr{Y}}$$

$$+60(\mathscr{Y}')^3\mathscr{Y}''f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}
+30(\mathscr{Y}')^2\mathscr{Y}'''f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}}
+60(\mathscr{Y}'')^2\mathscr{Y}'f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}}+20\mathscr{Y}'(\mathscr{Y}'')^2f_{tt\mathscr{Y}\mathscr{Y}}
+20(\mathscr{Y}')^2\mathscr{Y}''f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}}$$

$$+30\mathscr{Y}'\mathscr{Y}^{(4)}f_{t\mathscr{Y}\mathscr{Y}}+20\mathscr{Y}''\mathscr{Y}'''f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}}
+5(\mathscr{Y}')^5f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}+12(\mathscr{Y}'')^2\mathscr{Y}'f_{t\mathscr{Y}\mathscr{Y}}
+28\mathscr{Y}''\mathscr{Y}'''f_{t\mathscr{Y}\mathscr{Y}}+6\mathscr{Y}^{(5)}f_{t\mathscr{Y}}$$

$$+15(\mathscr{Y}')^4\mathscr{Y}''f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}
+10(\mathscr{Y}')^3\mathscr{Y}'''f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}
+30(\mathscr{Y}')^2(\mathscr{Y}'')^2f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}
+10\mathscr{Y}'''(\mathscr{Y}')^2f_{tt\mathscr{Y}\mathscr{Y}}
+10(\mathscr{Y}')^3\mathscr{Y}'''f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}$$

$$+15(\mathscr{Y}')^2\mathscr{Y}^{(4)}f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}}+20\mathscr{Y}'\mathscr{Y}''\mathscr{Y}'''f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}}
+9\mathscr{Y}^{(4)})\mathscr{Y}''f_{\mathscr{Y}\mathscr{Y}}+6\mathscr{Y}'\mathscr{Y}^{(5)}f_{\mathscr{Y}\mathscr{Y}}
+12(\mathscr{Y}'')^3f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}}+4\mathscr{Y}'\mathscr{Y}''\mathscr{Y}'''f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}$$

$$+4(\mathscr{Y}''')^2f_{\mathscr{Y}\mathscr{Y}}+3\mathscr{Y}''f_{tt\mathscr{Y}\mathscr{Y}}+3\mathscr{Y}''\mathscr{Y}'''f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}}
+5(\mathscr{Y}')^5f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}+12(\mathscr{Y}'')^2\mathscr{Y}'f_{t\mathscr{Y}\mathscr{Y}}
+28\mathscr{Y}''\mathscr{Y}'''f_{t\mathscr{Y}\mathscr{Y}}+6\mathscr{Y}^{(5)}f_{t\mathscr{Y}}$$

$$+15(\mathscr{Y}')^4\mathscr{Y}''f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}
+10(\mathscr{Y}')^3\mathscr{Y}'\mathscr{Y}''f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}}=\mathscr{Y}'''f_{t\mathscr{Y}\mathscr{Y}}
+3\mathscr{Y}''\mathscr{Y}'f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}}+3(\mathscr{Y}')^2\mathscr{Y}''f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}
+9\mathscr{Y}'\mathscr{Y}'''f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}}$$

$$
\begin{aligned}
&+3(\mathscr{Y}'')f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}}+6\mathscr{Y}'''f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}}+6\mathscr{Y}^{(4)}f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}}\\
&\quad+(\mathscr{Y}')^5f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}+(\mathscr{Y}')^6f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}\\
&\quad+\mathscr{Y}^{(6)}f_{\mathscr{Y}}\Big)h^6+O\big(h^6\big)
\end{aligned} \tag{17}
$$

$$
\begin{aligned}
\sum_{i=1}^{\mathscr{S}}b_i''''k_i&=\sum_{i=1}^{\mathscr{S}}b_i''''f+\sum_{i=1}^{\mathscr{S}}b_i''''c_i(f_t+f_{\mathscr{Y}}\mathscr{Y}')h\\
&+\frac{1}{2}\sum_{i=1}^{\mathscr{S}}b_i''''c_i^2\big(f_{tt}+2f_{t\mathscr{Y}}\mathscr{Y}'+f_{\mathscr{Y}}\mathscr{Y}''+f_{\mathscr{Y}\mathscr{Y}}(\mathscr{Y}')^2\big)h^2\\
&+\frac{1}{6}\sum_{i=1}^{\mathscr{S}}b_i''''c_i^3\big(f_{ttt}+(f')^3f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}}+3(\mathscr{Y}')^2f_{t\mathscr{Y}\mathscr{Y}}\\
&\quad+3\mathscr{Y}'\ \mathscr{Y}''f_{\mathscr{Y}\mathscr{Y}}+3\mathscr{Y}''f_{t\mathscr{Y}}+3\mathscr{Y}'f_{tt\mathscr{Y}}\\
&\quad+\mathscr{Y}'''f_{\mathscr{Y}}\big)h^3\\
&+\frac{1}{24}\sum_{i=1}^{\mathscr{S}}b_i''''c_i^4\Big(f_{tttt}+4\mathscr{Y}'f_{ttt\mathscr{Y}}+6(\mathscr{Y}')^2f_{tt\mathscr{Y}\mathscr{Y}}\\
&\quad+6\mathscr{Y}''\ \mathscr{Y}''f_{tt\mathscr{Y}}+4(\mathscr{Y}')^3f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}}\\
&\quad+12\mathscr{Y}'\mathscr{Y}''f_{t\mathscr{Y}\mathscr{Y}}+4\mathscr{Y}'''f_{\mathscr{Y}}\\
&+6(\mathscr{Y}')^2\mathscr{Y}''f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}}+4\mathscr{Y}'\mathscr{Y}'''f_{\mathscr{Y}\mathscr{Y}}\\
&\quad+3\mathscr{Y}''f_{\mathscr{Y}\mathscr{Y}}+(\mathscr{Y}')^4f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}+\mathscr{Y}^{(4)}f_{\mathscr{Y}}\Big)h^4\\
&+\frac{1}{120}\sum_{i=1}^{\mathscr{S}}b_i''''c_i^5\Big(f_{ttttt}+5\mathscr{Y}'f_{tttt\mathscr{Y}}\\
&\quad+10(\mathscr{Y}')^2f_{ttt\mathscr{Y}\mathscr{Y}}+10\mathscr{Y}''f_{ttt\mathscr{Y}}\\
&\quad+10(\mathscr{Y}')^3f_{tt\mathscr{Y}\mathscr{Y}\mathscr{Y}}+30\mathscr{Y}'\mathscr{Y}''f_{tt\mathscr{Y}\mathscr{Y}}\\
&+10\mathscr{Y}'''f_{tt\mathscr{Y}}+30(\mathscr{Y}')^2\mathscr{Y}''f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}}\\
&\quad+20\mathscr{Y}'\mathscr{Y}'''f_{t\mathscr{Y}\mathscr{Y}}+5(\mathscr{Y}')^4f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}\\
&\quad+12(\mathscr{Y}'')^2f_{t\mathscr{Y}\mathscr{Y}}+5\mathscr{Y}^{(4)}f_{t\mathscr{Y}}\\
&+10(\mathscr{Y}')^3\mathscr{Y}''f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}+10(\mathscr{Y}')^2\mathscr{Y}'''\\
&\quad+5\mathscr{Y}^{(4)}\mathscr{Y}'f_{\mathscr{Y}\mathscr{Y}}+12\mathscr{Y}'(\mathscr{Y}'')^3f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}}\\
&\quad+4\mathscr{Y}''\mathscr{Y}'''f_{\mathscr{Y}\mathscr{Y}}+3\mathscr{Y}''f_{t\mathscr{Y}\mathscr{Y}}\\
&+3\mathscr{Y}'\mathscr{Y}''f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}}+6\mathscr{Y}'''f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}}\\
&\quad+(\mathscr{Y}')^5f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}+\mathscr{Y}^{(5)}f_{-\mathscr{Y}}\Big)\ h^5\\
&+\frac{1}{720}\sum_{i=1}^{\mathscr{S}}b_i''''c_i^6\Big(f_{tttttt}+6\mathscr{Y}'f_{ttttt\mathscr{Y}}\\
&\quad+15(\mathscr{Y}')^2f_{tttt\mathscr{Y}\mathscr{Y}}+15\mathscr{Y}''f_{tttt\mathscr{Y}}\\
&\quad+20(\mathscr{Y}')^3f_{ttt\mathscr{Y}\mathscr{Y}\mathscr{Y}}+60\mathscr{Y}'\mathscr{Y}''f_{ttt\mathscr{Y}\mathscr{Y}}\\
&+20\mathscr{Y}'''f_{ttt\mathscr{Y}}+15(\mathscr{Y}')^4\mathscr{Y}''f_{tt\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}\\
&\quad+90\mathscr{Y}''(\mathscr{Y}')^2f_{tt\mathscr{Y}\mathscr{Y}\mathscr{Y}}+31\mathscr{Y}'\mathscr{Y}'''f_{tt\mathscr{Y}\mathscr{Y}}\\
&\quad+42(\mathscr{Y}')^2f_{tt\mathscr{Y}\mathscr{Y}}+15\mathscr{Y}^{(4)}f_{tt\mathscr{Y}}\\
&+60(\mathscr{Y}')^3\mathscr{Y}''f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}+30(\mathscr{Y}')^2\mathscr{Y}'''f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}}\\
&\quad+60(\mathscr{Y}'')^2\mathscr{Y}'f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}}+20\mathscr{Y}'(\mathscr{Y}'')^2f_{tt\mathscr{Y}\mathscr{Y}}\\
&\quad+20(\mathscr{Y}')^2\mathscr{Y}''f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}}\\
&+30\mathscr{Y}'\mathscr{Y}^{(4)}f_{t\mathscr{Y}\mathscr{Y}}+20\mathscr{Y}''\mathscr{Y}'''f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}}\\
&\quad+5(\mathscr{Y}')^5f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}+12(\mathscr{Y}'')^2\mathscr{Y}'f_{t\mathscr{Y}\mathscr{Y}}\\
&\quad+28\mathscr{Y}''\ \mathscr{Y}'''+f_{t\mathscr{Y}\mathscr{Y}}+6\mathscr{Y}^{(5)}f_{t\mathscr{Y}}\\
&+15(\mathscr{Y}')^4\mathscr{Y}''f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}\\
&\quad+10(\mathscr{Y}')^3\mathscr{Y}'''f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}+30(\mathscr{Y}')^2f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}\\
&\quad+10\mathscr{Y}'''(\mathscr{Y}')^2f_{tt\mathscr{Y}\mathscr{Y}\mathscr{Y}}\\
&\quad+10(\mathscr{Y}')^3\mathscr{Y}'''f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}\\
&+15(\mathscr{Y}')^2\mathscr{Y}^{(4)}f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}}+20\mathscr{Y}'\mathscr{Y}''\mathscr{Y}'''f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}}\\
&\quad+9\mathscr{Y}^{(4)})\mathscr{Y}''f_{\mathscr{Y}\mathscr{Y}}+6\mathscr{Y}'\mathscr{Y}^{(5)}f_{\mathscr{Y}\mathscr{Y}}\\
&\quad+12(\mathscr{Y}'')^3f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}}+4\mathscr{Y}'\mathscr{Y}''\mathscr{Y}'''f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}\\
&+4(\mathscr{Y}''')^2f_{\mathscr{Y}\mathscr{Y}}+3\mathscr{Y}''f_{tt\mathscr{Y}\mathscr{Y}}\\
&\quad+3\mathscr{Y}''\mathscr{Y}'''f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}}+5(\mathscr{Y}')^5f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}\\
&\quad+12(\mathscr{Y}'')^2\mathscr{Y}'f_{t\mathscr{Y}\mathscr{Y}}+28\mathscr{Y}''\mathscr{Y}'''f_{t\mathscr{Y}\mathscr{Y}}\\
&\quad+6\mathscr{Y}^{(5)}f_{t\mathscr{Y}}\\
&+15(\mathscr{Y}')^4\mathscr{Y}''f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}\\
&\quad+10(\mathscr{Y}')^3\mathscr{Y}'\mathscr{Y}''f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}}+3\mathscr{Y}'''f_{t\mathscr{Y}\mathscr{Y}}\\
&\quad+3\mathscr{Y}''\mathscr{Y}'f_{t\mathscr{Y}\mathscr{Y}}+3(\mathscr{Y}')^2\mathscr{Y}''f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}\\
&\quad+9\mathscr{Y}'\mathscr{Y}'''f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}}\\
&+3(\mathscr{Y}'')f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}}+6\mathscr{Y}'''f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}}+6\mathscr{Y}^{(4)}f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}}\\
&\quad+(\mathscr{Y}')^5f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}+(\mathscr{Y}')^6f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}\\
&\quad+\mathscr{Y}^{(6)}f_{\mathscr{Y}}\Big)h^6+O\big(h^7\big)
\end{aligned} \tag{18}
$$

$$
\begin{aligned}
\sum_{i=1}^{\mathscr{S}} b_i''''' k_i = {} & \sum_{i=1}^{\mathscr{S}} b_i''''' f + \sum_{i=1}^{\mathscr{S}} b_i''''' c_i (f_t + f_{\mathscr{Y}} \mathscr{Y}')h \\
& + \frac{1}{2} \sum_{i=1}^{\mathscr{S}} b_i''''' c_i^2 \big(f_{tt} + 22 f_{t\mathscr{Y}} \mathscr{Y}' + f_{\mathscr{Y}} \mathscr{Y}'' \\
& + f_{\mathscr{Y}\mathscr{Y}} (\mathscr{Y}')^2\big) h^2 \\
& + \frac{1}{6} \sum_{i=1}^{\mathscr{S}} b_i''''' c_i^3 \big(f_{ttt} + (f')^3 f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}} + 3(\mathscr{Y}')^2 f_{t\mathscr{Y}\mathscr{Y}} \\
& + 3\mathscr{Y}' \mathscr{Y}'' f_{\mathscr{Y}\mathscr{Y}} + 3\mathscr{Y}'' f_{t\mathscr{Y}} + 3\mathscr{Y}' f_{tt\mathscr{Y}} \\
& + \mathscr{Y}''' f_{\mathscr{Y}}\big) h^3 \\
& + \frac{1}{24} \sum_{i=1}^{\mathscr{S}} b_i''''' c_i^4 \Big(f_{tttt} + 4\mathscr{Y}' f_{ttt\mathscr{Y}} + 6(\mathscr{Y}')^2 f_{tt\mathscr{Y}\mathscr{Y}} \\
& + 6\mathscr{Y}'' f_{tt\mathscr{Y}} + 4(\mathscr{Y}')^3 f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}} + 12\mathscr{Y}' \mathscr{Y}'' f_{t\mathscr{Y}\mathscr{Y}} \\
& + 4\mathscr{Y}''' f_{t\mathscr{Y}} \\
& + 6(\mathscr{Y}')^2 \mathscr{Y}'' f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}} + 4\mathscr{Y}' \mathscr{Y}''' f_{\mathscr{Y}\mathscr{Y}} + 3\mathscr{Y}'' \\
& + f_{\mathscr{Y}\mathscr{Y}} + (\mathscr{Y}')^4 f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}} + \mathscr{Y}^{(4)} f_{\mathscr{Y}}\big) h^4 \\
& + \frac{1}{120} \sum_{i=1}^{\mathscr{S}} b_i''''' c_i^5 \Big(f_{ttttt} + 5\mathscr{Y}' f_{tttt\mathscr{Y}} \\
& + 10(\mathscr{Y}')^2 f_{ttt\mathscr{Y}\mathscr{Y}} + 10\mathscr{Y}'' f_{ttt\mathscr{Y}} \\
& + 10(\mathscr{Y}')^3 f_{tt\mathscr{Y}\mathscr{Y}\mathscr{Y}} + 30\mathscr{Y}' \mathscr{Y}'' f_{tt\mathscr{Y}\mathscr{Y}} \\
& + 10\mathscr{Y}''' f_{tt\mathscr{Y}} + 30(\mathscr{Y}')^2 \mathscr{Y}'' f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}} \\
& + 20\mathscr{Y}' \mathscr{Y}''' f_{t\mathscr{Y}\mathscr{Y}} + 5(\mathscr{Y}')^4 f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}} \\
& + 12(\mathscr{Y}'')^2 f_{t\mathscr{Y}\mathscr{Y}} + 5\mathscr{Y}^{(4)} f_{t\mathscr{Y}} \\
& + 10(\mathscr{Y}')^3 \mathscr{Y}'' f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}} + 10(\mathscr{Y}')^2 \mathscr{Y}''' f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}} \\
& + 5\mathscr{Y}^{(4)} \mathscr{Y}' f_{\mathscr{Y}\mathscr{Y}} + 12\mathscr{Y}' (\mathscr{Y}'')^2 f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}} \\
& + 4\mathscr{Y}'' \mathscr{Y}''' f_{\mathscr{Y}\mathscr{Y}} + 3\mathscr{Y}'' f_{t\mathscr{Y}\mathscr{Y}} \\
& + 3\mathscr{Y}' \mathscr{Y}'' f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}} + 6\mathscr{Y}''' f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}} \\
& + (\mathscr{Y}')^5 f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}} + \mathscr{Y}^{(5)} f_{\mathscr{Y}}\Big) h^5 \\
& + \frac{1}{720} \sum_{i=1}^{\mathscr{S}} b_i''''' c_i^6 \Big(f_{tttttt} + 6\mathscr{Y}' f_{ttttt\mathscr{Y}} \\
& + 15(\mathscr{Y}')^2 f_{tttt\mathscr{Y}\mathscr{Y}} + 15\mathscr{Y}'' f_{tttt\mathscr{Y}} \\
& + 20(\mathscr{Y}')^3 f_{ttt\mathscr{Y}\mathscr{Y}\mathscr{Y}} + 60\mathscr{Y}' \mathscr{Y}'' f_{ttt\mathscr{Y}\mathscr{Y}} \\
& + 20\mathscr{Y}''' f_{ttt\mathscr{Y}} + 15(\mathscr{Y}')^4 \mathscr{Y}'' f_{tt\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}} \\
& + 90\mathscr{Y}'' (\mathscr{Y}')^2 f_{tt\mathscr{Y}\mathscr{Y}\mathscr{Y}} + 31\mathscr{Y}' \mathscr{Y}''' f_{tt\mathscr{Y}\mathscr{Y}} \\
& + 42(\mathscr{Y}'')^2 f_{tt\mathscr{Y}\mathscr{Y}} + 15\mathscr{Y}^{(4)} f_{tt\mathscr{Y}} \\
& + 60(\mathscr{Y}')^3 \mathscr{Y}'' f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}} + 30(\mathscr{Y}')^2 \mathscr{Y}''' f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}} \\
& + 60(\mathscr{Y}'')^2 \mathscr{Y}' f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}} + 20\mathscr{Y}' (\mathscr{Y}'')^2 f_{tt\mathscr{Y}\mathscr{Y}} \\
& + 20(\mathscr{Y}')^2 \mathscr{Y}'' f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}} \\
& + 30\mathscr{Y}' \mathscr{Y}^{(4)} f_{t\mathscr{Y}\mathscr{Y}} + 20\mathscr{Y}'' \mathscr{Y}''' f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}} \\
& + 5(\mathscr{Y}')^5 f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}} + 12(\mathscr{Y}'')^2 \mathscr{Y}' f_{t\mathscr{Y}\mathscr{Y}} \\
& + 28\mathscr{Y}'' \mathscr{Y}''' f_{t\mathscr{Y}\mathscr{Y}} + 6\mathscr{Y}^{(5)} f_{t\mathscr{Y}} \\
& + 15(\mathscr{Y}')^4 \mathscr{Y}'' f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}} + 10(\mathscr{Y}')^3 \mathscr{Y}''' f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}} \\
& + 30(\mathscr{Y}')^2 (\mathscr{Y}'')^2 f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}} \\
& + 10\mathscr{Y}''' (\mathscr{Y}')^2 f_{tt\mathscr{Y}\mathscr{Y}\mathscr{Y}} + 10(\mathscr{Y}')^3 \mathscr{Y}''' f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}} \\
& + 15(\mathscr{Y}')^2 \mathscr{Y}^{(4)} f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}} + 20\mathscr{Y}' \mathscr{Y}'' \mathscr{Y}''' f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}} \\
& + 9\mathscr{Y}^{(4)}) \mathscr{Y}'' f_{\mathscr{Y}\mathscr{Y}} + 6\mathscr{Y}' \mathscr{Y}^{(5)} f_{\mathscr{Y}\mathscr{Y}} \\
& + 12(\mathscr{Y}'')^3 f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}} + 4\mathscr{Y}' \mathscr{Y}'' \mathscr{Y}''' f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}} \\
& + 4(\mathscr{Y}''')^2 f_{\mathscr{Y}\mathscr{Y}} + 3\mathscr{Y}'' f_{tt\mathscr{Y}\mathscr{Y}} + 3\mathscr{Y}'' \mathscr{Y}''' f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}} \\
& + 5(\mathscr{Y}')^5 f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}} + 12(\mathscr{Y}'')^2 \mathscr{Y}' f_{t\mathscr{Y}\mathscr{Y}} \\
& + 28\mathscr{Y}'' \mathscr{Y}''' f_{t\mathscr{Y}\mathscr{Y}} + 6\mathscr{Y}^{(5)} f_{t\mathscr{Y}} \\
& + 15(\mathscr{Y}')^4 \mathscr{Y}'' f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}} \\
& + 10(\mathscr{Y}')^3 \mathscr{Y}' \mathscr{Y}'' f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}} + 3\mathscr{Y}''' f_{t\mathscr{Y}\mathscr{Y}} \\
& + 3\mathscr{Y}'' \mathscr{Y}' f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}} + 3(\mathscr{Y}')^2 \mathscr{Y}'' f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}} \\
& + 9\mathscr{Y}' \mathscr{Y}''' f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}} \\
& + 3(\mathscr{Y}'') f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}} + 6\mathscr{Y}''' f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}} + 6\mathscr{Y}^{(4)} f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}} \\
& + (\mathscr{Y}')^5 f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}} + (\mathscr{Y}')^6 f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}} \\
& + \mathscr{Y}^{(6)} f_{\mathscr{Y}}\Big) h^6 + O(h^7)
\end{aligned} \tag{19}
$$

$$
\begin{aligned}
\sum_{i=1}^{\mathscr{S}} b_i'''''' k_i = {} & \sum_{i=1}^{\mathscr{S}} b_i'''''' f + \sum_{i=1}^{\mathscr{S}} b_i'''''' c_i (f + t + f_{\mathscr{Y}} \mathscr{Y}')h \\
& + \frac{1}{2} \sum_{i=1}^{\mathscr{S}} b_i'''''' c_i^2 \big(f_{tt} + 2 f_{t\mathscr{Y}} \mathscr{Y}' + f_{\mathscr{Y}} \mathscr{Y}'' \\
& + f_{\mathscr{Y}\mathscr{Y}} (\mathscr{Y}')^2\big) h^2
\end{aligned}
$$

$$+\frac{1}{6}\sum_{i=1}^{\mathscr{S}} b_i'''''' c_i^3\left(f_{ttt}+(f')^3 f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}}+3(\mathscr{Y}')^2 f_{t\mathscr{Y}\mathscr{Y}}\right.$$
$$\left.+3\mathscr{Y}'\mathscr{Y}'' f_{\mathscr{Y}\mathscr{Y}}+3\mathscr{Y}'' f_{t\mathscr{Y}}+3\mathscr{Y}' f_{tt\mathscr{Y}}+\mathscr{Y}''' f_{\mathscr{Y}}\right)h^3$$

$$+\frac{1}{24}\sum_{i=1}^{\mathscr{S}} b_i'''''' c_i^4\left(f_{tttt}+4\mathscr{Y}' f_{ttt\mathscr{Y}}+6(\mathscr{Y}')^2 f_{tt\mathscr{Y}\mathscr{Y}}\right.$$
$$+6\mathscr{Y}'' f_{tt\mathscr{Y}}4(\mathscr{Y}')^3 f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}}+12\mathscr{Y}'\mathscr{Y}'' f_{t\mathscr{Y}\mathscr{Y}}$$
$$+4\mathscr{Y}''' f_{t\mathscr{Y}}$$
$$+6(\mathscr{Y}')^2\mathscr{Y}'' f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}}+4\mathscr{Y}'\mathscr{Y}''' f_{\mathscr{Y}\mathscr{Y}}+3\mathscr{Y}'' f_{\mathscr{Y}\mathscr{Y}}$$
$$\left.+(\mathscr{Y}')^4 f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}+\mathscr{Y}^{(4)} f_{\mathscr{Y}}\right)h^4$$

$$+\frac{1}{120}\sum_{i=1}^{\mathscr{S}} b_i'''''' c_i^5\left(f_{ttttt}+5\mathscr{Y}' f_{tttt\mathscr{Y}}+10(\mathscr{Y}')^2 f_{ttt\mathscr{Y}\mathscr{Y}}\right.$$
$$+10\mathscr{Y}'' f_{ttt\mathscr{Y}}+10(\mathscr{Y}')^3 f_{tt\mathscr{Y}\mathscr{Y}\mathscr{Y}}$$
$$+30\mathscr{Y}'\mathscr{Y}'' f_{tt\mathscr{Y}\mathscr{Y}}$$

$$+10\mathscr{Y}''' f_{tt\mathscr{Y}}+30(\mathscr{Y}')^2\mathscr{Y}'' f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}}$$
$$+20\mathscr{Y}'\mathscr{Y}''' f_{t\mathscr{Y}\mathscr{Y}}+5(\mathscr{Y}')^4 f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}$$
$$+12(\mathscr{Y}'')^2 f_{t\mathscr{Y}\mathscr{Y}}+5\mathscr{Y}^{(4)} f_{\mathscr{Y}}$$
$$+10(\mathscr{Y}')^3\mathscr{Y}'' f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}+10(\mathscr{Y}')^2\mathscr{Y}''' f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}}$$
$$+5\mathscr{Y}^{(4)}\mathscr{Y}' f_{\mathscr{Y}\mathscr{Y}}+12\mathscr{Y}'(\mathscr{Y}'')^2 f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}}$$
$$+4\mathscr{Y}''\mathscr{Y}''' f_{\mathscr{Y}\mathscr{Y}}+3\mathscr{Y}'' f_{t\mathscr{Y}\mathscr{Y}}$$
$$+3\mathscr{Y}'\mathscr{Y}'' f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}}+6\mathscr{Y}''' f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}}+(\mathscr{Y}')^5 f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}$$
$$\left.+\mathscr{Y}^{(5)} f_{\mathscr{Y}}\right)h^5$$

$$+\frac{1}{720}\sum_{i=1}^{\mathscr{S}} b_i'''''' c_i^6\left(f_{tttttt}+6\mathscr{Y}' f_{ttttt\mathscr{Y}}+15(\mathscr{Y}')^2 f_{tttt\mathscr{Y}\mathscr{Y}}\right.$$
$$+20(\mathscr{Y}')^3 f_{ttt\mathscr{Y}\mathscr{Y}\mathscr{Y}}+60\mathscr{Y}'\mathscr{Y}'' f_{ttt\mathscr{Y}\mathscr{Y}}$$
$$+20\mathscr{Y}''' f_{ttt\mathscr{Y}}+15(\mathscr{Y}')^4\mathscr{Y}'' f_{tt\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}$$
$$+90\mathscr{Y}''(\mathscr{Y}')^2 f_{tt\mathscr{Y}\mathscr{Y}\mathscr{Y}}+31\mathscr{Y}'\mathscr{Y}''' f_{tt\mathscr{Y}\mathscr{Y}}$$
$$+42(\mathscr{Y}'')^2 f_{tt\mathscr{Y}\mathscr{Y}}+15\mathscr{Y}^{(4)} f_{tt\mathscr{Y}}$$
$$+60(\mathscr{Y}')^3\mathscr{Y}'' f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}+30(\mathscr{Y}')^2\mathscr{Y}''' f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}}$$
$$+60(\mathscr{Y}'')^2\mathscr{Y}' f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}}+20\mathscr{Y}'(\mathscr{Y}'')^2 f_{tt\mathscr{Y}\mathscr{Y}}$$
$$+20(\mathscr{Y}')^2\mathscr{Y}'' f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}}$$
$$+30\mathscr{Y}'\mathscr{Y}^{(4)} f_{t\mathscr{Y}\mathscr{Y}}+20\mathscr{Y}''\mathscr{Y}''' f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}}$$
$$+5(\mathscr{Y}')^5 f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}+12(\mathscr{Y}'')^2\mathscr{Y}' f_{t\mathscr{Y}\mathscr{Y}}$$
$$+28\mathscr{Y}''\mathscr{Y}''' f_{t\mathscr{Y}\mathscr{Y}}+6\mathscr{Y}^{(5)} f_{t\mathscr{Y}}$$
$$+15(\mathscr{Y}')^4\mathscr{Y}'' f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}+10(\mathscr{Y}')^3\mathscr{Y}''' f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}$$
$$+30(\mathscr{Y}')^2(\mathscr{Y}'')^2 f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}+10\mathscr{Y}'''(\mathscr{Y}')^2 f_{tt\mathscr{Y}\mathscr{Y}\mathscr{Y}}$$
$$+10(\mathscr{Y}')^3\mathscr{Y}''' f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}$$
$$+15(\mathscr{Y}')^2\mathscr{Y}^{(4)} f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}}+20\mathscr{Y}'\mathscr{Y}''\mathscr{Y}''' f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}}$$
$$+9\mathscr{Y}^{(4)})\mathscr{Y}'' f_{\mathscr{Y}\mathscr{Y}}+6\mathscr{Y}'\mathscr{Y}^{(5)} f_{\mathscr{Y}\mathscr{Y}}$$
$$+12(\mathscr{Y}'')^3 f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}}+4\mathscr{Y}'\mathscr{Y}''\mathscr{Y}''' f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}$$
$$+4(\mathscr{Y}''')^2 f_{\mathscr{Y}\mathscr{Y}}+3\mathscr{Y}'' f_{tt\mathscr{Y}\mathscr{Y}}+3\mathscr{Y}''\mathscr{Y}''' f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}}$$
$$+5(\mathscr{Y}')^5 f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}+12(\mathscr{Y}'')^2\mathscr{Y}' f_{t\mathscr{Y}\mathscr{Y}}$$
$$+28\mathscr{Y}''\mathscr{Y}''' f_{t\mathscr{Y}\mathscr{Y}}+6\mathscr{Y}^{(5)} f_{t\mathscr{Y}}$$
$$+15(\mathscr{Y}')^4\mathscr{Y}'' f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}+10(\mathscr{Y}')^3\mathscr{Y}'\mathscr{Y}'' f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}}$$
$$+3\mathscr{Y}''' f_{t\mathscr{Y}\mathscr{Y}}+3\mathscr{Y}''\mathscr{Y}' f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}}$$
$$+3(\mathscr{Y}')^2\mathscr{Y}'' f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}+9\mathscr{Y}'\mathscr{Y}''' f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}}$$
$$+3(\mathscr{Y}'') f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}}+6\mathscr{Y}''' f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}}+6\mathscr{Y}^{(4)} f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}}$$
$$+(\mathscr{Y}')^5 f_{t\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}+(\mathscr{Y}')^6 f_{\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}\mathscr{Y}}$$
$$\left.+\mathscr{Y}^{(6)} f_{\mathscr{Y}}\right)h^6+O\left(h^7\right) \tag{20}$$

The expression for local truncation errors in the solution, the derivatives up to the seventh-order $\mathscr{Y}^{(i)}; i=0,1,\ldots,6$ are

$$\mathscr{T}_{n+1}=h^7\left[\sum_{i=1}^{\mathscr{S}} b_i k_i-\left(\frac{1}{5040}F_1^{(7)}+\frac{1}{40320}F_1^{(8)}\right.\right.$$
$$\left.\left.+\frac{1}{362880}F_1^{(9)}+\frac{1}{3628800}F_1^{(10)}+\cdots\cdots\right)\right], \tag{21}$$

$$\mathscr{T}'_{(n+1)} = h^8\left[\sum_{i=1}^{\mathscr{S}} b'_i k_i - \left(\frac{1}{720}F_1^{(7)} + \frac{1}{5040}F_1^{(8)} + \frac{1}{40320}F_1^{(9)} + \frac{1}{362880}F_1^{(10)} + \cdots\cdots\right)\right], \tag{22}$$

$$\mathscr{T}''_{(n+1)} = h^9\left[\sum_{i=1}^{\mathscr{S}} b''_i k_i - \left(\frac{1}{120}F_1^{(7)} + \frac{1}{720}F_1^{(8)} + \frac{1}{5040}F_1^{(9)} + \frac{1}{40320}F_1^{(10)} + \cdots\cdots\right)\right], \tag{23}$$

$$\mathscr{T}'''_{(n+1)} = h^{10}\left[\sum_{i=1}^{\mathscr{S}} b'''_i k_i - \left(\frac{1}{24}F_1^{(7)} + \frac{1}{120}F_1^{(8)} + \frac{1}{720}F_1^{(9)} + \frac{1}{5040}F_1^{(10)} + \cdots\cdots\right)\right], \tag{24}$$

$$\mathscr{T}''''_{(n+1)} = h^{11}\left[\sum_{i=1}^{\mathscr{S}} b''''_i k_i - \left(\frac{1}{6}F_1^{(7)} + \frac{1}{24}F_1^{(8)} + \frac{1}{120}F_1^{(9)} + \frac{1}{720}F_1^{(10)} + \cdots\cdots\right)\right], \tag{25}$$

$$\mathscr{T}'''''_{(n+1)} = h^{12}\left[\sum_{i=1}^{\mathscr{S}} b'''''_i k_i - \left(\frac{1}{2}F_1^{(7)} + \frac{1}{6}F_1^{(8)} + \frac{1}{24}F_1^{(9)} + \frac{1}{120}F_1^{(10)} + \cdots\cdots\right)\right], \tag{26}$$

$$\mathscr{T}''''''_{(n+1)} = h^{13}\left[\sum_{i=1}^{\mathscr{S}} b''''''_i k_i - \left(F_1^{(7)} + \frac{1}{2}F_1^{(8)} + \frac{1}{6}F_1^{(9)} + \frac{1}{24}F_1^{(10)} + \frac{1}{120}F_1^{(11)}\cdots\cdots\right)\right], \tag{27}$$

3.2. Order conditions

To determine the order conditions of the numerical integrators indicated by Equations 3−11, RKM method formula is expanded using the approach of Taylor's series expansion using Maple software. Hence the following order conditions are given:

$$\mathscr{Y} \;:\; \sum b_i = \frac{1}{5040}, \sum b_i c_i = \frac{1}{40320}, \sum b_i c_i^2 = \frac{1}{181440}, \sum b_i c_i^3 = \frac{1}{604800}, \tag{28}$$

$$\mathscr{Y}' : \sum b'_i = \frac{1}{720}, \sum b'_i c_i = \frac{1}{5040}, \sum b'_i c_i^2 = \frac{1}{20160} = \sum b'_i c_i^3 = \frac{1}{60480} = \sum b'_i c_i^4 = \frac{1}{151200}. \quad (29)$$

$$\mathscr{Y}'' : \sum b''_i = \frac{1}{120}, \sum b''_i c_i = \frac{1}{720}, \sum b''_i c_i^2 = \frac{1}{2520}, \sum b''_i c_i^3 = \frac{1}{13440}, \sum b''_i c_i^4 = \frac{1}{15120}. \sum b''_i c_i^5 = \frac{1}{30240}. \quad (30)$$

$$\mathscr{Y}''' : \sum b'''_i = \frac{1}{24}, \sum b'''_i c_i = \frac{1}{120}, \sum b'''_i c_i^2 = \frac{1}{360} \sum b'''_i c_i^3 = \frac{1}{840} \sum b'''_i c_i^4 = \frac{1}{1680}, \sum b''''_i c_i^5 = \frac{1}{3024}, \sum b'''_i c_i^6 = \frac{1}{5040}. \quad (31)$$

$$\mathscr{Y}'''' : \sum b''''_i = \frac{1}{6}, \sum b''''_i c_i = \frac{1}{24}, \sum b''''_i c_i^2 = \frac{1}{60}, \sum b''''_i c_i^3 = \frac{1}{120}, \sum b''''_i c_i^4 = \frac{1}{210}, \sum b'''''_i c_i^5 = \frac{1}{336}, \sum b''''_i c_i^6 = \frac{1}{5040}, \sum b''''_i c_i^6 = \frac{1}{720}. \quad (32)$$

$$\mathscr{Y}''''' : \sum b'''''_i = \frac{1}{2}, \sum b'''''_i c_i = \frac{1}{6}, \sum b'''''_i c_i^2 = \frac{1}{12}, \sum b'''''_i c_i^3 = \frac{1}{20}, \sum b'''''_i c_i^4 = \frac{1}{30}, \sum b''''''_i c_i^5 = \frac{1}{42}, \sum b'''''_i c_i^6 = \frac{1}{56}, \sum b'''''_i c_i^7 = \frac{1}{72}, \sum b'''''_i c_i^8 = \frac{1}{90}. \quad (33)$$

$$\mathscr{Y}'''''' : \sum b''''''_i = 1, \sum b''''''_i c_i = \frac{1}{2}, \sum b''''''_i c_i^2 = \frac{1}{3}, \sum b''''''_i c_i^3 = \frac{1}{4}, \sum b''''''_i c_i^4 = \frac{1}{5}, \sum b'''''''_i c_i^5 = \frac{1}{6},$$

Table 2
The Butcher Tableau for the RKM Method of Four Stages and Fifth Order.

0	0	0	0	0
$\frac{1}{2}$	$\frac{-37}{160}-\frac{1}{240}\sqrt{15}$	0	0	0
$\frac{1}{2}-\frac{1}{10}\sqrt{15}$	$\frac{1}{10}$	$\frac{1}{10}$	0	0
$\frac{1}{2}+\frac{1}{10}\sqrt{15}$	$\frac{29}{100}-\frac{1}{75}\sqrt{15}$	$\frac{19}{100}+\frac{1}{50}\sqrt{15}$	$\frac{1}{10}$	0
	0	$\frac{1}{103680}$	$\frac{11}{103680}+\frac{71}{2592000}\sqrt{15}$	$\frac{11}{103680}-\frac{71}{2592000}\sqrt{15}$
	0	$\frac{1}{8640}$	$\frac{11}{17280}+\frac{71}{432000}\sqrt{15}$	$\frac{11}{17280}+\frac{71}{432000}\sqrt{15}$
	0	$\frac{1}{864}$	$\frac{31}{8640}+\frac{1}{1080}\sqrt{15}$	$\frac{31}{8640}+\frac{1}{1080}\sqrt{15}$
	0	$\frac{1}{108}$	$\frac{7}{432}+\frac{1}{240}\sqrt{15}$	$\frac{7}{432}-\frac{1}{240}\sqrt{15}$
	0	$\frac{1}{18}$	$\frac{1}{18}+\frac{1}{72}\sqrt{15}$	$\frac{1}{18}-\frac{1}{72}\sqrt{15}$
	0	$\frac{2}{9}$	$\frac{3}{36}+\frac{1}{36}\sqrt{15}$	$\frac{3}{36}-\frac{1}{36}\sqrt{15}$
	0	$\frac{4}{9}$	$\frac{5}{18}$	$\frac{5}{18}$

$$\cdot \sum b_i^{''''''} c_i^6 = \frac{1}{7}, \sum b_i^{''''''} c_i^7 = \frac{1}{8}, \sum b_i^{''''''} c_i^8 = \frac{1}{9}, \sum b_i^{''''''} c_i^6 = \frac{1}{10}, \sum_{i=2,\, j=1}^{5,4} \sum_{j<i} b_i^{''''''} a_{ij} = \frac{1}{40320},$$

$$\cdot \sum_{i=2,\, j=1}^{5,4} \sum_{j<i} b_i^{''''''} a_{ij} c_i = \frac{1}{45360}, \sum_{i=3,\, j=2}^{5,4} \sum_{j<i} b_i^{''''''} a_{ij} c_i = \frac{1}{362880}, \sum_{i=2,\, j=1}^{5,4} \sum_{j<i} b_i^{''''''} a_{ij} c_i^2 = \frac{1}{50400},$$

$$\cdot \sum_{i=2,\, j=1}^{5,4} \sum_{j<i} b_i^{''''''} a_{ij} c_i^2 = \frac{1}{1814400}, \tag{34}$$

3.3. Derivation of RKM methods

To derive proposed RKM methods using the algebraic conditions (28–34) with the following assumption:

Table 4
Number of stages versus order for RK, RKN, RKD and the recently proposed RKM methods.

NO. of Stages Order	3	4	5	6
NO. of Stages of RK method	3	4	6	7
NO. of Stages of RKN method	2	3	4	5
NO. of Stages of RKD method	2	3	3	4
NO. of Stages of RKM method	2	3	4	4

$$b_i = \frac{(1-c_i)^6}{6!} b_i^{''''''}, b_i' = \frac{(1-c_i)^5}{5!} b_i^{''''''}, b_i'' = \frac{(1-c_i)^4}{4!} b_i^{''''''},$$

$$b_i''' = \frac{(1-c_i)^3}{3!} b_i^{''''''}, b_i'''' = \frac{(1-c_i)^2}{2!} b_i^{''''''},$$
$$b_i^{'''''} = (1-c_i) b_i^{''''''},$$

For $i = 1, \ldots\ldots, \mathscr{S}$,

The parameters of RKM method c_i, a_{ij}, b_i, b_i', b_i'', b_i''', b_i'''', $b_i^{'''''}$ and $b_i^{''''''}$ for $i,j = 1,2,\ldots\ldots,\mathscr{S}$. for Four-stage fifth-order and five-stage sixth-order RKM integrators have been evaluated and the Butcher tableaus of these integrators are shown in Tables 2 and 3 respectively (see Table 4).

Table 3
The Butcher Tableau for the RKM Method of Five Stages and Sixth Order.

0	0	0	0	0	0
1	$\frac{1}{2}$	0	0	0	0
$\frac{1}{2} - \frac{1}{14}\sqrt{21}$	$\frac{-1}{2}$	0	0	0	0
$\frac{1}{2} + \frac{1}{14}\sqrt{21}$	$\frac{1}{2}$	$\frac{-1}{2}$	$\frac{1}{2}$	0	0
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{-1}{2}$	$\frac{1}{2}$	$\frac{-7167}{14336}$	0
	$\frac{1}{14400}$	0	$\frac{19}{259200} + \frac{29}{1814400}\sqrt{21}$	$\frac{19}{259200} - \frac{29}{1814400}\sqrt{21}$	$\frac{181}{138240}$
	$\frac{1}{2400}$	0	$\frac{19}{43200} + \frac{29}{302400}\sqrt{21}$	$\frac{19}{43200} - \frac{29}{302400}\sqrt{15}$	$\frac{1}{76800}$
	$\frac{1}{480}$	0	$\frac{23}{8640} + \frac{29}{1728}\sqrt{21}$	$\frac{23}{8640} - \frac{29}{1728}\sqrt{21}$	$\frac{1}{1080}$
	$\frac{1}{120}$	0	$\frac{7}{540} + \frac{1}{360}\sqrt{21}$	$\frac{7}{540} - \frac{1}{360}\sqrt{21}$	$\frac{1}{135}$
	$\frac{1}{40}$	0	$\frac{7}{144} + \frac{7}{720}\sqrt{21}$	$\frac{7}{144} - \frac{7}{720}\sqrt{21}$	$\frac{2}{45}$
	$\frac{1}{20}$	0	$\frac{49}{360} + \frac{7}{360}\sqrt{21}$	$\frac{49}{360} - \frac{7}{360}\sqrt{21}$	$\frac{8}{45}$
	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{49}{180}$	$\frac{49}{180}$	$\frac{16}{45}$

4. Stability of the method

4.1. Zero stability of the method

Definition 4.1 The method is said to be zero stable if it satisfied $-1 < \xi \le 1$ [15].

Zero-stability is an important tool for proving the stability and convergence of linear multistep methods. We can rewrite equations (3)−(9) as follows:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} w_{n+1} \\ hw'_{n+1} \\ h^2 w''_{n+1} \\ h^3 w'''_{n+1} \\ h^4 w''''_{n+1} \\ h^5 w'''''_{n+1} \\ h^6 w''''''_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 & \frac{1}{2} & \frac{1}{6} & \frac{1}{24} & \frac{1}{120} & \frac{1}{720} \\ 0 & 1 & 1 & \frac{1}{2} & \frac{1}{6} & \frac{1}{24} & \frac{1}{120} \\ 0 & 0 & 1 & 1 & \frac{1}{2} & \frac{1}{6} & \frac{1}{24} \\ 0 & 0 & 0 & 1 & 1 & \frac{1}{2} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} w_n \\ hw'_n \\ h^2 w''_n \\ h^3 w'''_n \\ h^4 w''''_n \\ h^5 w'''''_n \\ h^6 w''''''_n \end{pmatrix} \tag{35}$$

$$\rho(\xi) = det|I\xi - A| = \begin{pmatrix} \xi-1 & -1 & \frac{-1}{2} & \frac{-1}{6} & \frac{-1}{24} & \frac{-1}{120} & \frac{-1}{720} \\ 0 & \xi-1 & -1 & \frac{-1}{2} & \frac{-1}{6} & \frac{-1}{24} & \frac{-1}{120} \\ 0 & 0 & \xi-1 & -1 & \frac{-1}{2} & \frac{-1}{6} & \frac{-1}{24} \\ 0 & 0 & 0 & \xi-1 & -1 & \frac{-1}{2} & \frac{-1}{6} \\ 0 & 0 & 0 & 0 & \xi-1 & -1 & \frac{-1}{2} \\ 0 & 0 & 0 & 0 & 0 & \xi-1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \xi-1 \end{pmatrix} \tag{36}$$

Thus the characteristic polynomial is

$$\rho(\xi) = (\xi-1)^7, \tag{37}$$

Hence, the method is zero-stable since the roots are $\xi = 1, 1, 1, 1, 1, 1, 1$ are less or equal to one.

4.2. Absolute stability of the method

Definition 4.2 The method is said to be absolutely stable for a given roots if all the roots lies within the unit circle [15,16] has studied the absolute stability for RKD method and compared the stability regions for RKD and RKT methods, In the same way we have studied the absolute stability for the RKM method and we apply this to the test problem:

$$\omega^{(7)} = -\chi^7\omega \tag{38}$$

Now, consider formulas Equations (3)–(9), which can be written for the test problem as follows:

We can rewrite equations (3)–(9) in the following matrix notation after multiplying them by h, h^2, h^3, h^4, h^5 and h^6 respectively and using Equation (38):

$$f_{n+1} = \begin{pmatrix} 1 & 1 & \frac{1}{2} & \frac{1}{6} & \frac{1}{24} & \frac{1}{120} & \frac{1}{720} \\ 0 & 1 & 1 & \frac{1}{2} & \frac{1}{6} & \frac{1}{24} & \frac{1}{120} \\ 0 & 0 & 1 & 1 & \frac{1}{2} & \frac{1}{6} & \frac{1}{24} \\ 0 & 0 & 0 & 1 & 1 & \frac{1}{2} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} f_n + x^7h^7 \begin{pmatrix} b_1 & b_2 & b_3 & \dots & b_{\mathscr{S}} \\ b'_1 & b'_2 & b'_3 & \dots & b'_{\mathscr{S}} \\ b''_1 & b''_2 & b''_3 & \dots & b''_{\mathscr{S}} \\ b'''_1 & b'''_2 & b'''_3 & \dots & b'''_{\mathscr{S}} \\ b''''_1 & b''''_2 & b''''_3 & \dots & b''''_{\mathscr{S}} \\ b'''''_1 & b'''''_2 & b'''''_3 & \dots & b'''''_{\mathscr{S}} \\ b''''''_1 & b''''''_2 & b''''''_3 & \dots & b''''''_{\mathscr{S}} \end{pmatrix} \begin{pmatrix} W_1 \\ W_2 \\ W_3 \\ W_4 \\ \vdots \\ \vdots \\ W_S \end{pmatrix},$$

and the Equation (27) as,

$$\begin{pmatrix} W_1 \\ W_2 \\ W_3 \\ W_4 \\ \vdots \\ W_S \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & c_2 & \frac{c_2^2}{2} & \frac{c_2^3}{6} & \frac{c_2^4}{24} & \frac{c_2^5}{120} & \frac{c_2^6}{720} \\ 1 & c_3 & \frac{c_3^2}{2} & \frac{c_3^3}{6} & \frac{c_3^4}{24} & \frac{c_3^5}{120} & \frac{c_3^6}{720} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & c_{\mathscr{S}-1} & \frac{c_{\mathscr{S}-1}^2}{2} & \frac{c_{\mathscr{S}-1}^3}{6} & \frac{c_{\mathscr{S}-1}^4}{24} & \frac{c_{\mathscr{S}-1}^5}{120} & \frac{c_{\mathscr{S}-1}^6}{720} \\ 1 & c_{\mathscr{S}} & \frac{c_{\mathscr{S}}^2}{2} & \frac{c_{\mathscr{S}}^3}{6} & \frac{c_{\mathscr{S}}^4}{24} & \frac{c_{\mathscr{S}}^5}{120} & \frac{c_{\mathscr{S}}^6}{720} \end{pmatrix} f_{n+H} \begin{pmatrix} 0 & \cdots & & & & & \\ a_{21} & 0 & \cdots & & & & \\ a_{31} & a_{32} & 0 & \cdots & & & \\ a_{41} & a_{42} & a_{43} & 0 & \cdots & & \\ \vdots & \vdots & \vdots & \vdots & \ddots & \cdots & \\ a_{\mathscr{S}1} & a_{\mathscr{S}2} & a_{\mathscr{S}3} & a_{\mathscr{S}4} & \cdots & a_{\mathscr{S}\mathscr{S}-1} & 0 \end{pmatrix}, \begin{pmatrix} W_1 \\ W_2 \\ W_3 \\ W_4 \\ \vdots \\ W_S \end{pmatrix},$$

Hence,

$f_{n+1} = R(H) f_n,$

$$R(H) = \begin{pmatrix} 1+b^T V & 1+b^T M & \frac{1}{2}+b^T L & \frac{1}{6}+b^T K & \frac{1}{24}+b^T E & \frac{1}{120}+b^T Y \\ b'^T V & 1+b'^T M & 1+b'^T L & \frac{1}{2}+b'^T K & \frac{1}{6}+b'^T E & \frac{1}{24}+b'^T Y \\ b''^T V & b''^T M & 1+b''^T L & 1+b''^T K & \frac{1}{2}+b''^T E & \frac{1}{6}+b''^T Y \\ b'''^T V & b'''^T M & b'''^T L & 1+b'''^T K & 1+b'''^T E & \frac{1}{2}+b'''^T Y \\ b''''^T V & b''''^T M & b''''^T L & b''''^T K & 1+b''''^T E & 1+b''''^T Y \\ b'''''^T V & b'''''^T M & b'''''^T L & b'''''^T K & b'''''^T E & 1+b'''''^T Y \end{pmatrix}, \tag{39}$$

Where, $H = \chi^7 h^7 = (\chi h)^7$, $HN^{-1}e = V$, $HN^{-1}c = M$, $HN^{-1}d = L$, $HN^{-1}p = K$, $HN^{-1}r = E_{,}^{-1}q = Y$.

$$e = (1,1,1,1,1,1,\ldots,1)^T,\ c = (0, c_1, c_2, c_3, c_4, \ldots, c_{\mathscr{S}})^T,\ d = \left(0, \frac{c_2^2}{2}, \frac{c_3^2}{2}, \frac{c_4^2}{2}, \ldots, \frac{c_{\mathscr{S}}^2}{2}\right)^T,$$

$$p = \left(0, \frac{c_2^3}{6}, \frac{c_3^3}{6}, \frac{c_4^3}{6}, \ldots, \frac{c_{\mathscr{S}}^3}{6}\right)^T,\ r = \left(0, \frac{c_2^4}{24}, \frac{c_3^4}{24}, \frac{c_4^4}{24}, \ldots, \frac{c_{\mathscr{S}}^4}{24}\right)^T,\ q = \left(0, \frac{c_2^5}{120}, \frac{c_3^5}{120}, \ldots, \frac{c_{\mathscr{S}}^5}{120}\right)^T.$$

With,

$N^{-1} = I - HA,$

$$A = \begin{pmatrix} 0 & \cdots & & & & & 0 \\ a_{21} & 0 & \cdots & \cdots & & & 0 \\ a_{31} & a_{32} & 0 & \cdots & \cdots & & 0 \\ a_{41} & a_{42} & a_{43} & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & & \ddots & & 0 \\ \vdots & \vdots & \vdots & & & & 0 \\ a_{\mathscr{S}1} & a_{\mathscr{S}2} & a_{\mathscr{S}3} & a_{\mathscr{S}4} & \cdots & a_{\mathscr{S}\mathscr{S}-1} & 0 \end{pmatrix}, B = \begin{pmatrix} b_1 & b_2 & b_3 & \cdots & \cdots & b_{\mathscr{S}} \\ b'_1 & b'_2 & b'_3 & \cdots & \cdots & b'_{\mathscr{S}} \\ b''_1 & b''_2 & b''_3 & \cdots & \cdots & b''_{\mathscr{S}} \\ b'''_1 & b'''_2 & b'''_3 & \cdots & \cdots & b'''_{\mathscr{S}} \\ b''''_1 & b''''_2 & b''''_3 & \cdots & \cdots & b''''_{\mathscr{S}} \\ b'''''_1 & b'''''_2 & b'''''_3 & \cdots & \cdots & b'''''_{\mathscr{S}} \\ b''''''_1 & b''''''_2 & b''''''_3 & \cdots & \cdots & b''''''_{\mathscr{S}} \end{pmatrix}$$

And,

$$
\mathrm{NCF}_n=\begin{pmatrix} W_1\\ W_2\\ W_3\\ W_4\\ .\\ .\\ Ws\end{pmatrix},\ f_n=\begin{pmatrix} w_n\\ hw_n'\\ h^2w_n''\\ h^3w_n'''\\ h^4w_n''''\\ h^5w_n'''''\\ h^6w_n''''''\end{pmatrix},\quad C=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ 1 & c_2 & \frac{c_2^2}{2} & \frac{c_2^3}{6} & \frac{c_2^4}{24} & \frac{c_2^5}{120}\\ 1 & c_3 & \frac{c_3^2}{2} & \frac{c_3^3}{6} & \frac{c_3^4}{24} & \frac{c_3^5}{120}\\ \vdots & \vdots & & \vdots & & \vdots\\ \vdots & \vdots & & \ddots & & \vdots\\ \vdots & \vdots & & \ddots & & \vdots\\ 1 & c_3 & \frac{c_{\mathscr{S}}^2}{2} & \frac{c_{\mathscr{S}}^3}{6} & \frac{c_{\mathscr{S}}^4}{24} & \frac{c_{\mathscr{S}}^5}{120}\end{pmatrix},
$$

The stability function associated with this method is given by,

$\Theta(\xi,\mathrm{H})=|\xi I-RH|,$

Where R(H) a rational function of H, and defined in Equation (28) is a stability matrix and its characteristic equation can be written as:

$$
\Theta(\xi,\mathrm{H})=\lambda_0(H)\xi^7+\lambda_1(H)\xi^6+\lambda_2(H)\xi^5+\lambda_3(H)\xi^4 \\ +\lambda_4(H)\xi^3+\lambda_5(H)\xi^2+\lambda_6(H)\xi.
$$

5. Implementations

In this section, we imply the proposed methods for solving four problems and their numerical results introduced in Fig. 2.

Example 5.1 (Linear ODE)

$\mathscr{Y}^{(7)}(t)=\sin(t,\ )\qquad 0<t\leq 1.$

subject to ICs: $\mathscr{Y}^{(i)}(0)=(-1)^i$ *for* $i=0,2,4,6$ *and* $\mathscr{Y}^{(i)}(0)=0$ *for* $i=1,3,5$

The exact solution is $\mathscr{Y}^{(t)}=\cos(t)$.

Example 5.2 (Non Constant Coefficients ODE)

$\mathscr{Y}^{(7)}(t)=\left(128t^7+1344t^5+3360t^3+1680t\right)\mathscr{Y}(t);$ $0<t\leq 1.$

subject to ICs: $\mathscr{Y}^{(i)}(0)=0; i=1,3,5, \mathscr{Y}(0)=1,$ $\mathscr{Y}''(0)=2,\ \mathscr{Y}^{(4)}(0)=12,\ \mathscr{Y}^{(6)}(0)=120.$

The exact solution is $\mathscr{Y}(t)=e^{t^2}$

Example 5.3 (Non Linear ODE)

$\mathscr{Y}^{(7)}(t)=-5040\,\mathscr{Y}^8(t),\qquad 0<t\leq 1.$

subject to ICs: $\mathscr{Y}(0)=-\mathscr{Y}'(0)=1,3\mathscr{Y}''(0)=-\mathscr{Y}'''(0)=6,5\mathscr{Y}^{(4)}(0)=-\mathscr{Y}^{(5)}(0)=120,$ $\mathscr{Y}^{(6)}(0)=720.$

The exact solution is $\mathscr{Y}(t)=\frac{1}{1+t}$

Example 5.4 (Linear ODE with Relatively Long Interval)

$\mathscr{Y}^{(7)}(t)=0.0000001e^{-\frac{t}{10}},\ 0<t\leq 1.$

subject to ICs: $\mathscr{Y}(0)=1, \mathscr{Y}^{(i)}(0)=(-0,1)^i; i=1,2,\ldots,6.$

The exact solution is $\mathscr{Y}(t)=e^{-\frac{t}{10}}$

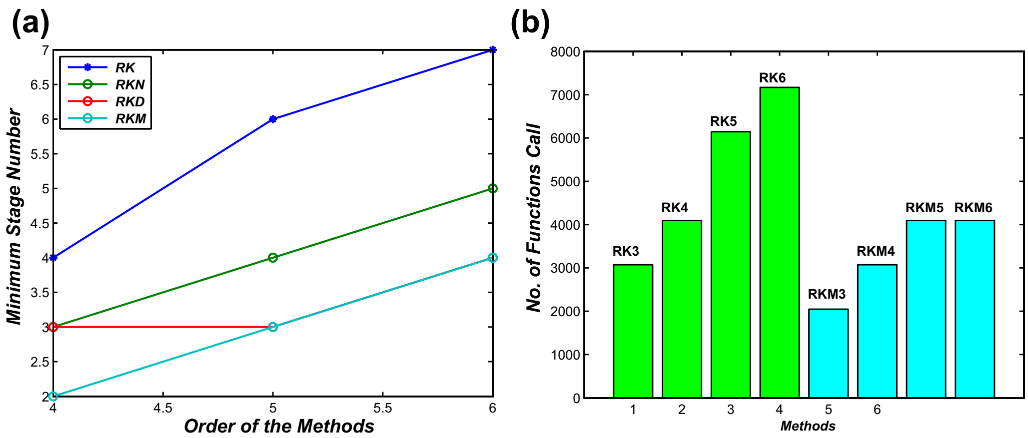

Fig. 1. (a) Minimum stage number versus method order for RK, RKN, and the recently proposed RKM method. (b) Number of function calls for RK and RKM methods with orders 4,5,6.

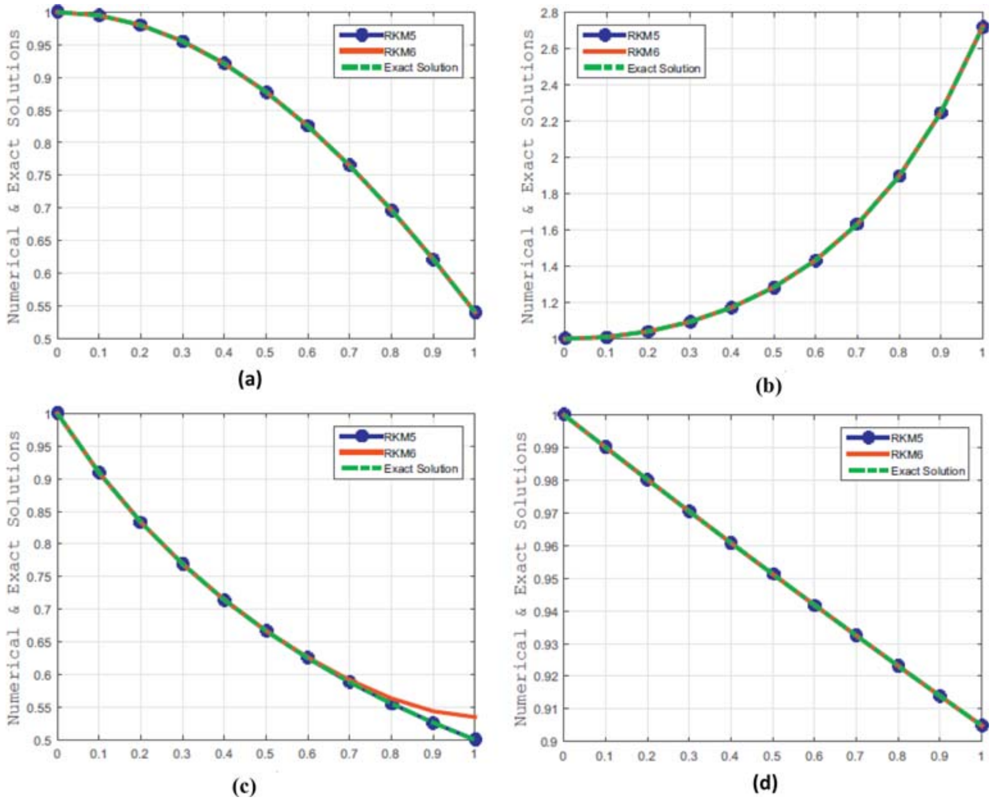

Fig. 2. Comparison of numerical solutions with exact solution for examples (a)1, (b)2, (c)3, (d)4.

6. Discussion and conclusion

In this study, we have derived the algebraic equations of order conditions for direct integrators of RKM for class of seventh-order ODES. The approach of the derivation of the new method is based on Taylor expansion. The objective of this work is to establish direct explicit integrators of RK type for solving two classes of fifth-order ODEs. For this purpose, we have generalized the integrators RK, RKN, RKD, RKT and RKFD which are used for solving class of first-, second-, third-, fourth- and fifth-order ODES. We have derived four-stage, fifth-order and five-stage, sixth-order RKM methods. Numerical results using the proposed methods have been compared with analytical solutions in Fig. 2 which shows that the numerical results of the four test problems are more efficient and accurate as well-known existing methods. The new integrates are more efficient in implementation as they require less function evaluations. As such, these methods are more cost effective, in terms of computation time, than existing methods (see Figures 1,2 and Tables 3,4).

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