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Approximating fixed points in modular spaces

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Abstract

A generic two theorems for the two step iterative sequence of multivalued mappings are proved in a complete convex real modular space, and then cite some corollaries that are special cases of these theorems.

Keywords

Multivalued mappings; Fixed points; Iterative sequences; Uniformly convex real modular spaces.

Cover Page Footnote

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1. Introduction and preliminaries

Modular spaces are extensions of Lebesgue, Riesz, and Orlicz spaces of integrable functions. A general theory of modular linear spaces was founded by Nakano in Ref. [1], where he developed a spectral theory in semi ordered linear spaces (vector lattices) and established the integral representation for projections acting in his modular space; Nakano's modulars on real linear spaces are convex functionals. Nonconvex modulars and the corresponding modular linear spaces were constructed by Musielak and Orlicz [2]. Orlicz spaces and modular linear spaces have already become classical tools in modern nonlinear functional analysis. Recent work indicates that modular metric space fixed point results are well adapted to certain types of differential equations [3]. Finally, we refer to Ref. [4] for a detailed study of nonlinear superposition operators on modular metric spaces of functions [5−8]. In the formulation given by Khamsi [9]:

Definition 1.1

Let μ be a linear space over $F(= R\ or\ \mathbb{C})$. A function $\gamma : \mu \rightarrow [0, \infty]$ is called modular if

(i) $\gamma\ (v) = 0$ if and only if $v=0$,
(ii) $\gamma(\alpha v) = \alpha(v)$ for F with $|\alpha| = 1$, for all, $v \in \mu$
(iii) $\gamma(\alpha v+\ \beta u) \leq \gamma(v)+\gamma(u)$ Iff$\alpha, \beta \geq 0$,$\alpha+\ \beta = 1$ for all $u,\ v \in M$.

If (iii) replaced by

(iii) '$\gamma(\alpha v+\ \beta u) \leq\ \alpha\gamma(v)+\beta\gamma(u)$, for$\alpha, \beta \geq 0$, $\alpha+\beta = 1$, for all u, v $\in$M, then γ is called convex modular.

Definition 1.2 [1]

A modular γ defines a corresponding modular space, μ_γ, given by

$\mu_\gamma = \{v \in \mu : \gamma(\alpha v) \rightarrow 0 \text{ whenever } \alpha \rightarrow 0\}$

Many works can be found in Ref. [10−12].

Definition 1.3 [13]

The γ-ball, $B_r(u)$ centered at $u \in \mu_\gamma$ with radius $r > 0$ as$B_r(u) = \{v \in \mu_\gamma; \gamma(u-\ v) < r\}$.

The class of all γ-balls in a modular space μ_γ generates a topology which makes μ_γ Hausdorff topological linear space. Every γ-ball is a convex set, therefore every modular space is locally convex Hausdorff topological vector space [6].

Definition 1.4 [6] Let M_γ be a modular space.

(a) A sequence $\{v_n\} \subset M_\gamma$ is said to be γ-convergent to$v \in M_\gamma$ and write $v_n \xrightarrow{\gamma} v$if $\gamma(v_n - v) \rightarrow 0$ as n$\rightarrow \infty$.
(b) A sequence $\{v_n\}$ is called γ Cauchy whenever $\gamma(v_n - v_m) \rightarrow 0$ as $n, m \rightarrow \infty$.
(c) $M_{\gamma s}$ is called γ complete if any γ Cauchy sequence in $M_{\gamma s}$ is γ convergent.
(d) A subset $B \subset M_\gamma$ is called γ closed if for any sequence $\{v_n\} \subset B$ is γ convergent to a point in B
(e) A γ closed subset $B \subset M_\gamma$ is called γ compact if any sequence $\{v_n\} \subset B$ has a γ convergent subsequence.
(f) A subset $B \subset M_\gamma$is said to be γ bounded if $daim_\gamma(B) < \infty$, where daimγ(B)=sup {γ(v−u);v,u∈B} is called the γ diameter of B.
(g) The distance between $v \in M_\gamma$ and $B \subset M_\gamma$ is $\gamma(v - B) = \inf\{\gamma(v-\ u); u \in B\}$.

Definition 1.5 [14]

Let μ_γ be a modular space, and A, B are two non − empty subsets of μ_γ. Let $H_\gamma(A, B)$ denotes the Hausdorff distance of A and B that is defined as the following: $H_\gamma(A,\ B) = max\ \{sup_{aJA}\gamma(a - B\ ), sup_{bJB}\ \gamma(b - A)\}$.

Lemma 1.6 [15]

Let $T : A \rightarrow 2^A$ be a modular space, $A_n B_n$ sequences in $CB(\mu_\gamma)$ Then we can choose a_n in A_n , $b_n$in B_n such that

$$\gamma\ (a_n - b_n\ ) = H_\gamma\ (A_n\ , B_n\ ) + \epsilon_n, \lim_{n\rightarrow\infty} \epsilon_n = 0 \tag{1}$$

Let A be a non-empty subset of μ_γ, Abed and Abduljabbar [15,16] introduced the following iterative sequence of two-step type for multivalued mapping $T : A \rightarrow 2^A$ $u_0 \in A$ and $\{u_n\} \subset A$ is defined by

$$u_{n+1} \in (1 - a_n)u_n +\ a_n T v_n$$
$$v_n \in (1 - \beta_n)u_n + \beta_n T u_n, \vee n\ \geq\ 0 \tag{2}$$

or

$$\begin{aligned} u_{n+1} &= (1 - a_n)u_n\ + a_n\mu_n, \mu_n\ \in\ T v_n, \quad n \geq 0 \\ v_n &= (\ 1 - \beta_n)u_n + \beta_n\xi_n, \xi_n \in\ T u_n\ n\ \geq 0 \end{aligned} \tag{3}$$

The following iterative sequence of multivalued mappings $S, T : A \rightarrow 2^A$ $u_0 \in A$ is defined by

$$\begin{aligned} u_{n+1} &\in (1 - a_n)u_n + a_n S v_n \\ v_n &\in (1 - \beta_n)u_n + \beta_n T u_n, \vee n\ \geq\ 0 \end{aligned} \tag{4}$$

or

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$$u_{n+1} = (1-\alpha_n)u_n + \alpha_n\zeta_n,\ \zeta_n \in Sv_n,\ \forall\ n \geq 0$$
$$v_n = \in (1-\beta_n)\, u_n + \beta_n\xi_n, \quad (5)$$
$$\xi_n \in Tu_n$$

In this article, It is assumed that the iterative sequences (1.3) and (1.5) converge. Moreover, it converges to a fixed point of T. Also, some results that are special cases of these theorems are presented. Here, μ_γ is a complete convex real modular space (shortly, *CCRMS*).

2. A fixed point theorem for multivalued mappings

We begin with the following

Theorem (2.1)

Let μ_γ be a *CCRMS*, $\emptyset \neq A \subset M$, A be convex $T: A \rightarrow CB(A)$, and $\{u_n\}$ as in (3) satisfying $\lim_{n\to\infty} \inf \alpha_n > 0 \ni \{u_n\}$ converges to p. Suppose that $\exists$ $\alpha, \beta, \mu, \delta > 0,\ \beta < 1 \ni$ for all n sufficiently large

$$H_y(Tu_n, Tv_n) \leq \alpha\gamma(u_n - \mu_n) + \beta\gamma(u_n - \xi_n) \quad (6)$$

$$H(T_n, Tu_n) \leq \alpha\gamma(u_n - p) + \mu d_y(u_n, Tu_n) + \delta d_y(p, Tu_n) + \beta\max\{d_\gamma(p, T_p), d_\gamma(u_n, TP)\} \quad (7)$$

where $\alpha_n + \beta_n = 1$for all n. Then p is a fixed point of T

Proof:

Use condition (3) $u_{n+1} = (1-a_n)u_n + a_n\mu_n,\ \mu_n \in Tv_n,\quad n \geq 0\ q < 1.$ Since$\gamma(\xi_n - \mu_n) \leq H_\gamma(T u_n, T v_n) + \epsilon_n$

Also, then $\lim_{n\to\infty} u_n = 0$ Which means that $\lim_{n\to\infty} \mu_n = p$

By conditions (1) and (6) we have:

$$\gamma(u_n - p) + \gamma(u_n - \xi_n) + \alpha\gamma(u_n - p) + \gamma(u_n - \xi_n) + \delta\gamma(p - \xi_n) + \beta\max\{d_\gamma(p, Tp), \gamma(u_n - p), d_\gamma(p, Tp)\}$$
$$\leq \alpha\gamma(u_n - \mu_n) + \beta\gamma(u_n - \xi_n) + \epsilon_n$$

Which implies that $\lim_{n\to\infty} \xi_n = p$. Using (7) to have

$$d_\gamma(p, Tp) \leq \gamma(u_n - p) + d_\gamma(u_n, Tu_n) + H_\gamma(Tp, Tu_n) \leq \gamma(u_n - p) + \gamma(u_n - \xi_n) + \alpha\gamma(u_n - p) + \mu d_\gamma(u_n, Tu_n) + \delta d_\gamma(p, Tu_n) + \beta\max\{d_\gamma(p, Tp), d_\gamma(u_n, Tp)\} \leq \gamma(u_n - p) + \gamma(u_n - \xi_n) + \alpha\gamma(u_n - p) + \gamma(u_n - \xi_n) + \delta\gamma(p - \xi_n) +$$
$$\beta\max\{d_\gamma(p, Tp),\ \gamma(u_n - p), d_\gamma(p, Tp)\}$$

Taking the limit as $n \to \infty$ yields. $d_\gamma(p, T p) \leq \beta\, d_\gamma(p, T p)$ Which implies that $p \in Tp$. $p \in Tp$

Corollary (2.2)

Let M_γ be a *CCRMS* , $\emptyset \neq A \subset M_\gamma$, A be convex $T: A \rightarrow CB(A)$ satisfying

$$H_\gamma(Tu, Tv) \leq q\max\{k\gamma(u - v),\ d_\gamma(v, Tv), d_\gamma(u, Tv) + d_\gamma(v, Tu)\quad (8)$$

where $q < 1$.

å $\lim_{n\to\infty} \inf \alpha_n > 0$ *and* $\lim_{n\to\infty} \beta_n = 0$,

converges to p , then p is a fixed point of T .

Proof:

It is sufficient to show that T satisfies conditions (6) and (7). From the condition (8), we get

$$H_\gamma(Tu_n, Tv_n) \leq q\max\{k\gamma(u_n - v_n), d_\gamma(u_n, Tu_n) + d_\gamma(v_n, Tv_n), d_\gamma(u_n, Tv_n) + d_\gamma(v_n, Tu_n)\} \quad (9)$$

From condition (3),

$$v_n = (1 - \beta_n)u_n + \beta_n\xi_n, \qquad \xi_n \in Tu_n \quad \text{for all } n$$

We have

$$\gamma(u_n - v_n) = \beta_n\gamma(u_n - \xi_n)$$
$$d_\gamma(v_n, Tu_n) \leq \gamma(v_n - \xi_n) = \gamma((1-\beta_n)u_n + \beta_n\xi_n - \xi_n)$$
$$= \gamma((1-\beta_n)u_n - (1-\beta_n)\xi_n)$$
$$= (1-\beta_n)\gamma(u_n - \xi_n),$$
$$d_\gamma(v_n, Tv_n) \leq \gamma(v_n, \mu_n),\quad \mu_n \in Tv_n$$
$$\leq \gamma(u_n - v_n) + \gamma(u_n - \mu_n) \leq \beta_n\gamma(u_n - \xi_n) + \gamma(u_n - \mu_n)$$
$$d_\gamma(u_n, Tu_n) \leq \gamma(u_n - \xi_n), \xi_n \in Tu_n$$

And $d_\gamma(u_n, Tv_n) \leq \gamma(u_n - \mu_n)$. Substituting into (9) gives

$$H_\gamma(Tu_n, Tv_n) \leq q\max\{k\beta_n\gamma(u_n - \xi_n), \gamma(u_n - \xi_n) + \beta_n\gamma(u_n - \xi_n) + \gamma(u_n - \mu_n), \gamma(u_n - \mu_n) + (1-\beta_n)\gamma(u_n - \xi_n)\}$$
$$\leq q\gamma(u_n - \mu_n) + \max\{kq\beta_n, q(1 + \beta_n)\}\gamma(u_n - \xi_n)$$

R Since $\beta_n \to 0$ as $n \to \infty$. So, there exists n_0 large enough to make $\max\{kq\beta_n, q(1+\beta_n)\} < 1$ And (6) is satisfied. Again from (8), we obtain

$H_\gamma(T u_n, Tp) \leq q\max\{k\gamma(u_n - p), d_\gamma(u_n, Tu_n) + d_\gamma(p, Tp), d_\gamma(u_n, Tp) + d_\gamma(p, Tu_n)\} \leq q\, k\, \gamma(u_n - p) + qd_\gamma(u_n, Tu_n) + q(p, Tu_n) + q\max\{d_\gamma(p, Tp), d_\gamma(u_n, Tp)\}$. It is clear that *if* $\alpha = qk$, $\mu = \delta = \beta = q < 1$ then (8) is satisfied.

Corollary (2.3)

Let M_γ be a *CCRMS*, $\emptyset \neq A \subset M_\gamma$ A be convex $T: A \rightarrow CB(A)$ satisfying

$$H_\gamma(Tu,Tv) \le \max\left\{\gamma(u-v),\frac{d_\gamma(u,Tu)+d_\gamma(v,Tv)}{2},\ \frac{d_\gamma(u,Tv)+d_\gamma(v,Tu)}{2}\right\} \tag{10}$$

For all, u,v *in* A if there exists $u_0 \in A$ such that $\{u_n\}$ in condition (3) satisfying $0<\alpha_n$, $\beta_n \le 1$, $\liminf_{n\to\infty}\alpha_n > 0$, $\limsup_{n\to\infty}\beta_n < 1$ and condition (1) converges to p, then p is a fixed point of T.

Proof: From (10), we obtain

$$H_\gamma(Tu_n,Tv_n) \le \max\left\{\gamma(u_n-v_n),\frac{d_\gamma(u_n,Tu_n)+d_\gamma(v_n,Tv_n)}{2},\frac{d_\gamma(u_n,Tv_n)+d_\gamma(v_n,Tu_n)}{2}\right\} \tag{11}$$

But from the condition (3), we have

$\gamma(u_n-v_n)=\beta_n\gamma(u_n-\xi_n)$,
$d_\gamma(v_n,Tu_n) \le (1-\beta_n)\gamma(u_n-\xi_n)$,
$d_\gamma(v_n,Tv_n) \le \beta_n\gamma(u_n-\xi_n)+\gamma(u_n-\mu_n)$
$d_\gamma(u_n,Tu_n) \le \gamma(u_n-\xi_n)$ *and* $d_\gamma(u_n,Tv_n) \le \gamma(u_n-\mu_n)$

Substituting into (11) yields:

$$\begin{aligned}H_\gamma(Tu_n,Tv_n) &\le \max\left\{\beta_n\gamma(u_n-\xi_n),\frac{\gamma(u_n-\xi_n)+\beta_n\gamma(u_n-\xi_n)+\gamma(u_n-\mu_n)}{2},\frac{\gamma(u_n-\mu_n)+(1-\beta_n)\gamma(u_n-\xi_n)}{2}\right\}\\ &\le \max\left\{\beta_n,\frac{1+\beta_n}{2}\right\}\gamma(u_n-\xi_n)+\frac{1}{2}\gamma(u_n-\mu_n)\end{aligned}$$

Since $\limsup_{n\to\infty}\beta_n < 1$ then we can choose n_0 large enough to make $\max\left\{\beta_n,\ \frac{1+\beta_n}{2}\right\} < 1$ and condition (6) is satisfied. From (*10*)

$$\begin{aligned}&H_\gamma(Tu_n,Tp) \le \max\Big\{\gamma(u_n-p),\\ &\frac{d_\gamma(u_n,Tu_n)+d_\gamma(p,Tp)}{2},\\ &\frac{d_\gamma(u_n,p)+d_\gamma(p,Tu_n)}{2}\Big\}\\ &\le \gamma(u_n-p)+\frac{d_\gamma(u_n,Tu_n)}{2}+\frac{d_\gamma(p,Tu_n)}{2}\\ &+\frac{1}{2}\max\{d_\gamma(p,Tp),d_\gamma(u_n,p)\}\end{aligned}$$

so, $\alpha=1, \mu=\delta=\beta=\frac{1}{2}$ and condition (2.2) is satisfied.

Corollary(2.4):

Let M_γ be a *CCRMS*, $\varnothing \ne A \subset M_\gamma$, A be; convex, $T: A \to CB(A)$ satisfying

$$H_\gamma(Tu,Tv) \le q\max\left\{\gamma(u-v),\frac{d_\gamma(v,Tv)[1+d_\gamma(u,Tu)]}{1+\gamma(u-v)},\frac{d_\gamma(u,Tv)[1+d_\gamma(u,Tu)+d_\gamma(v,Tu)]}{2[1+\gamma(u-v)]}\right\} \tag{12}$$

$\exists\ u,v$ *in* A where $0<q<1$. If there exists u_0 in $A \ni \{u_n\}$ satisfying condition (3) and (1) for $\alpha_n+\beta_n=1$ $\liminf_{n\to\infty}\alpha_n>0$ and $\lim\beta_n=0$, converges to p, then p is a fixed point of T.

Proof: From (12), we obtain

Form condition (3),

$$v_n-\mu_n=(1-\beta_n)u_n+\beta_n\xi_n-\beta_n\mu_n-(1-\beta_n)\mu_n$$

So,

$$\begin{aligned}\gamma(v_n-\mu_n) &\le (1-\beta_n)\gamma(u_n-\mu_n)+\beta_n\gamma(\xi_n-\mu_n)\\ &\le (1-\beta_n)\gamma(u_n-\mu_n)+\beta_n[\gamma(u_n-\mu_n)\\ &\quad+\gamma(u_n-\xi_n)]\\ &=\gamma(u_n-\mu_n)+\beta_n\gamma(u_n-\xi_n)\end{aligned}$$

Also, from condition (3), $\gamma(u_{n+1}-u_n)=\alpha_n\gamma(u_n-\mu_n)$

$$H_\gamma(Tu_n,Tv_n) \le q\max\left\{\gamma(u_n-v_n),\ \frac{d_\gamma(v_n,Tv_n)[1+d_\gamma(u_n,Tu_n)]}{1+\gamma(u_n-v_n)},\ \frac{d_\gamma(u_n,Tv_n)[1+d_\gamma(u_n,Tu_n)+d_\gamma(v_n,Tu_n)]}{2[1+\gamma(u_n-v_n)]}\right\}$$

$$\le q\max\left\{\gamma(u_n-v_n),\frac{\gamma(u_n-\mu_n)[1+\gamma(u_n-\xi_n)]}{1+\gamma(u_n-\xi_n)},\frac{\gamma(u_n-\mu_n)[1+\gamma(u_n-\xi_n)+\gamma(v_n-\xi_n)]}{2[1+\gamma(u_n-\xi_n)]}\right\}$$

Since u_n is convergent $\lim_{n\to\infty}\gamma(u_{n+1}-u_n)=0$and from $\lim_{n\to\infty}\inf\alpha_n>0$ Yields $\lim_{n\to\infty}\gamma(u_n-\mu_n)=0$. Therefore, for all n sufficiently large,

$\gamma(u_n-\mu_n)+\beta_n\,\gamma(u_n-\xi_n)\le 1+\beta_n\,\gamma(u_n-\xi_n).$

Thus, for all n sufficiently large and from inequalities in the proof of corollary (6), we have

$$\frac{\gamma(v_n-\mu_n)[1+\gamma(u_n-\xi_n)]}{1+\beta_n\,\gamma(u_n-\xi_n)} \le \gamma(v_n-\mu_n)+\frac{\gamma(v_n-\mu_n)\gamma(u_n-\xi_n)}{1+\beta_n\gamma(u_n-\xi_n)} \le \gamma(v_n-\mu_n)+\gamma(u_n-\xi_n)$$
$$\le \gamma(u_n-\mu_n)+\beta_n\,\gamma(u_n-\xi_n)+\gamma(u_n-\xi_n)=(1-\beta_n)\gamma(u_n-\xi_n)+\gamma(u_n-\mu_n)$$

Since

$v_n-\xi_n=(1-\beta_n)u_n+\beta_n\xi_n-\beta_n\xi_n-(1-\beta_n)\xi_n$

So,

$\gamma(v_n-\xi_n)=(1-\beta_n)\,\gamma(u_n-\xi_n)$

Since

$$\frac{\gamma(u_n-\mu_n)[1+\gamma(u_n-\xi_n)+\gamma(v_n-\xi_n)]}{2[1+\beta_n\gamma(u_n-\xi_n)]} = \frac{\gamma(u_n-\mu_n)[1+(2-\beta_n)\,\gamma(u_n-\xi_n)}{2[1+\beta_n\gamma(u_n-\xi_n)]} \le \frac{1}{2}[\gamma(u_n-\mu_n)+(2-\beta_n)\gamma(u_n-\xi_n)]$$

Then for all n sufficiently large, we get:

$$H_\gamma(Tu_n,Tv_n)\le q\max\{\beta_n\gamma(u_n-\xi_n),\ (1+\beta_n)\gamma(u_n-\xi_n)+\gamma(u_n-\mu_n),\ \frac{1}{2}\left[\gamma(u_n-\mu_n)+(2-\beta_n)\gamma(u_n-\xi_n)\right\}$$
$$\le \max\{q\beta_n,\ q(1+\beta_n),q(2-\beta_n)/2\}\ \gamma(u_n-\xi_n) + q\,\gamma(u_n-\mu_n)\}$$

And (6) is satisfied, since $\lim_{n\to\infty}\beta_n=0$ Again from (12), we get

$$H_\gamma(Tu_n,Tp)\le q\max\left\{\gamma\left(u_n-p\right),\ \frac{d_\gamma(p,Tp)[1+\gamma(u_n-\xi_n)]}{1+\gamma(u_n-p)},\ \frac{d_\gamma(u_n,Tp)[1+\gamma(u_n-\xi_n)+\gamma(p-\xi_n)]}{2[1+\gamma(u_n-p)]}\right\}\le q\gamma(u_n-p)$$
$$+q\max\left\{\frac{1+\gamma(u_n-\xi_n)}{1+\gamma(u_n-p)},\ \frac{1+\gamma(u_n-\xi_n)+\gamma(p-\xi_n)}{2[1+\gamma(u_n-p)]}\right\}$$
$$\max\{d_\gamma(p,Tp),d_\gamma(u_n,Tp)\}$$

Since the condition (6) is satisfied

$$\gamma(u_n-\xi_n)\le\gamma(u_n-\mu_n)+\gamma(\mu_n-\xi_n)$$
$$\le\gamma(u_n-\mu_n)+H_\gamma(Tu_n,Tv_n)+\epsilon_n$$
$$\le\gamma(u_n-\mu_n)+\alpha\,\gamma(u_n-\mu_n)+\beta\,\gamma(u_n-\xi_n)$$
$$+\ \epsilon_n\alpha_n,\beta<0\ and\ \beta<1$$

since $\lim_{n\to\infty}\gamma(u_n - \mu_n) = 0$, we have $\limsup_{n\to\infty}\gamma(u_n-\xi_n) \le \beta \limsup_{n\to\infty}\gamma(u_n-\xi_n)$, since $0\le\beta\le1$, which implies that $\lim_{n\to\infty}\gamma(u_n-\xi_n)=0$ *since* $\gamma(p-\xi_n)\le\gamma(p-u_n)+\gamma(u_n-\xi_n)$, it follows that

$$\lim_{n\to\infty}\max\left\{\frac{1+\gamma(u_n-\xi_n)}{1+\gamma(u_n-p)},\frac{1+\gamma(u_n-\xi_n)+\gamma(p-\xi_n)}{2[1+\gamma(u_n-p)]}\right\} =\max\left\{1,\frac{1}{2}\right\}= 1.$$

3. Common fixed point for a pair of mappings

We replace the condition (6), (7)and (1) by taking $\xi_n\in Tu_n$ and$\mu_n\in Sv_n$, $n\in N$

$$\gamma(\xi_n-\mu_n)\le H_\gamma(Tu_n,Sv_n)+\epsilon_n \text{ with } \lim_{n\to\infty}\epsilon_n=0 \tag{13}$$

$$H_\gamma(Tu_n,Sv_n)\le\alpha\,\gamma(u_n-\mu_n)+\beta\,\gamma(u_n-\xi_n) \tag{14}$$

$$H_\gamma(Sp,Tu_n)\le\alpha\gamma(u_n-p)+\gamma d_\gamma(u_n,Tu_n) + \delta d_\gamma(p,Tu_n) + \beta\max\{d_\gamma(p,Sp),d_\gamma(u_n,Sp)\} \tag{15}$$

Also, assume that

$$H_\gamma(Sp,Tp)\le\beta[d_\gamma(p,Tp)+d_\gamma(p,Sp)] \tag{16}$$

Theorem (3.1)

Let μ_γ be a *CCRMS*, $\varnothing\ne A\subset\mu_\gamma$ and A be convex $S, T: A\to CB(A)$. Suppose that $\{u_n\}$ as in (5) converges to point p, where $\alpha_n+\beta_n=1$, $\forall n$and $\liminf_{n\to\infty}\alpha_n>0$ $\{\xi_n\},\{\mu_n\}$ satisfying (13). If for all n sufficiently large, S and T satisfy (14), (15) and (16). Then p is a common fixed point for S and T

Proof: Use condition (5) $u_{n+1}=(1-\alpha_n)u_n+\alpha_n\mu_n$, $\mu_n\in Sv_n$. We have $\gamma(u_{n+1}-u_n)=\alpha_n\gamma(u_n-\mu_n)$ since $\lim_{n\to\infty}u_n=p$then $\lim_{n\to\infty}(u_{n+1}-u_n)=0$.

Also, since $\liminf_{n\to\infty}\alpha_n=0$ then which means that $\lim_{n\to\infty}\mu_n=p$. Using condition (13) and (14), we have:

$$\gamma(\xi_n-\mu_n)\le H_\gamma(Tu_n,Sv_n)+\epsilon_n \le\alpha\gamma(u_n-\mu_n)+\beta\gamma(u_n-\xi_n)+\epsilon_n$$

Taking limit as $n\to\infty$ yields, $\lim_{n\to\infty}\gamma(\xi_n-p)\le\beta\lim_{n\to\infty}\gamma(p-\xi_n)$, which implies that $\lim_{n\to\infty}\xi_n=p$

Using condition (15) to have:

$$\begin{aligned}d_\gamma(p,Sp)&\le\gamma(u_n-p)+d_\gamma(u_n,Tu_n)+H_\gamma(Sp,Tu_n)\\&\le\gamma(u_n-p)+\gamma(u_n-\xi_n)+\alpha\gamma(u_n-p)+\gamma d_\gamma(u_n,Tu_n)+\delta d_\gamma(p,Tu_n)+\beta\max\{d_\gamma(p,Sp),d_\gamma(u_n,Sp)\}\\&\le\gamma(u_n-p)+\gamma(u_n-\xi_n)+\alpha\gamma(u_n-p)+\mu(u_n-\xi_n)+\delta\gamma(p-\xi_n)+\beta\max\{d_\gamma(p,Sp),\gamma(u_n-p)+d_\gamma(p,Sp)\}\\&\le(1+\alpha)\gamma(u_n-p)+(1+\gamma)\gamma(u_n-\xi_n)+\delta\gamma(p-\xi_n)+\beta\max\{d_\gamma(p,Sp),\gamma(u_n-p)+d_\gamma(p,Sp)\}\end{aligned}$$

Taking limit as $n\to\infty$ yields $d_\gamma(p,Sp)\le\beta\,d_\gamma(p,Sp)$ which implies that $p\in Sp$. To show that p is also a fixed point of *T*, using condition (16)

$$d_\gamma(p,Tp)\le H_\gamma(Sp,Tp)\le\beta[d_\gamma(p,Sp)+d_\gamma(p,Tp)] =\beta d_\gamma(p,Tp)$$

So, p must be an element of Tp

Corollary (3.2)

Let μ_γ be a *CCRMS*, $\varnothing\ne A\subset\mu_\gamma$ be convex $S,T: A\to CB(A)$ satisfying

$$H_\gamma(Tu,Sv)\le q\max\{k\gamma(u-v),d_\gamma(u,Tu)+d_\gamma(v,Sv),\,d_\gamma(u,Sv)+d_\gamma(v,Tu)\} \tag{17}$$

$\exists\, u,v$ in A where $K\ge0$ *and* $0<q<1$. If there exists u_0 in $A\ni\{u_n\}$ satisfying (5) and (13) for $\alpha_n+\beta_n=1$ $\liminf_{n\to\infty}\alpha_n>0$ *and* $\lim_{n\to\infty}\beta_n=0$, converges to p then p is a fixed point of T.

Proof: It is sufficient to show that T satisfies conditions (14), (15) and (16) from (17), we obtain

$$H_\gamma(Tu_n,Sv_n)\le q\max\{k\gamma(u_n-v_n), d_\gamma(u_n,Tu_n)+d_\gamma(v_n,Sv_n), d_\gamma(u_n,Sv_n)+d_\gamma(v_n,Tu_n)\} \tag{18}$$

From condition (3), $v_n=(1-\beta_n)u_n+\beta_n\xi_n$, $\xi_n\in Tu_n$ for all n. We have $\gamma(u_n-v_n)=\beta\gamma(u_n-\xi_n)$,

$$\begin{aligned}d_\gamma(v_n,Tu_n)&\le\gamma(v_n-\xi_n)=\gamma((1-\beta_n)u_n+\beta_n\xi_n-\xi_n)\\&=\gamma((1-\beta_n)u_n-(1-\beta_n)\xi_n)\\&=(1-\beta_n)\,\gamma(u_n-\xi_n), d_\gamma(v_n,Sv_n)\\&\le\gamma(v_n-\mu_n)\quad,\mu_n\in Sv_n\\&\le\gamma(u_n-v_n)+\gamma(u_n-\mu_n)\\&\le\beta_n\gamma(u_n-\xi_n)+\gamma(u_n-\mu_n)d_\gamma(u_n,Tu_n)\\&\le\gamma(u_n-\xi_n),\xi_n\in Tu_n\end{aligned}$$

And $d_\gamma(u_n, Sv_n) \le \gamma(u_n - \mu_n)$.
Substituting into (18) gives

$$H_\gamma(Tu_n,Sv_n) \le qmax\{k\beta_n\gamma(u_n-\xi_n), \gamma(u_n-\xi_n)+\beta_n\gamma(u_n-\xi_n)+\gamma(u_n-\mu_n), \gamma(u_n-\mu_n)+(1-\beta_n)\gamma(u_n-\xi_n)\} \le q\gamma(u_n-\mu_n)+\max\{kq\beta_n, q(1+\beta_n)\}\gamma(u_n-\xi_n)\}$$

because $\beta_n \to \infty$ $as\ n \to \infty$ So, there exists n_0 large enough to make $\max\{kq\beta_n, q(1+\beta_n)\} < 1$ and (14) is satisfied. Again from (17)

$$H_\gamma(Tu_n,Sp) \le qmax\{k\gamma(u_n-p), d_\gamma(u_n,Tu_n)+d_\gamma(p,Sp), d_\gamma(u_n,Sp)+d_\gamma(p,Tu_n)\} \le qk\,\gamma(u_n-p)+qd_\gamma(u_n,Tu_n) +qd_\gamma(p,Tu_n)+qmax\,\{d_\gamma(p,Sp),d_\gamma(u_n,Sp)\}$$

It is clear that, if $\alpha = qk$, $\mu=\delta=\beta= q<1$, then (15) is satisfied from (17)

$$H_\gamma(Tp, Sp) \le qmax\{k\gamma(p-p), d_\gamma(p,Tp)+d_\gamma(p,Sp), d_\gamma(p,Sp)+d_\gamma(p,Tp)\} \le q\max\{0, d_\gamma(p,Tp), d_\gamma(p,Sp)\}$$

and (16) is satisfied with $\beta = q < 1$.

Corollary (3.3)

Let μ_γ be a *CCRMS*, $\varnothing \ne A \subset \mu_\gamma$ be convex $S,T: A \to CB(A)$ satisfying

$$H_\gamma(Tu,Sv) \le max\left\{\gamma(u-v), \frac{d_\gamma(u_n,Tu_n)+d_\gamma(v_n,Sv_n)}{2}, \frac{d_\gamma(u_n,Sv_n)+d_\gamma(v_n,Tu_n)}{2}\right\} \quad (19)$$

$\exists\ u,v\ in\ A$.If there exists a point $u_0 \in A \ni \{u_n\}$in (5) satisfying $\alpha_n+\beta_n=1$, $\forall n \lim_{n\to\infty}\inf\alpha_n>0$, $\lim_{n\to\infty}\sup\beta_n<1$ and (13), converges to p, then p is a fixed point of T.

Proof: from (19), we get

$$H_\gamma(Tu_n, Sv_n) \le \max\left\{\gamma(u_n-v_n), \frac{d_\gamma(u_n,Tu_n)+d_\gamma(v_n,Sv_n)}{2}, \frac{d_\gamma(u_n,Sv_n)+d_\gamma(v_n,Tu_n)}{2}\right\} \quad (20)$$

But from the condition (5), we have

$$\gamma(u_n-v_n)=\beta_n\,\gamma(u_n-\xi_n),$$
$$d_\gamma(v_n,Tu_n) \le (1-\beta_n)\gamma(u_n-\xi_n),$$
$$d_\gamma(v_n,Su_n) \le \beta_n\gamma(u_n-\xi_n)+\gamma(u_n-\mu_n),$$

$d_\gamma(u_n, Tu_n) \le \gamma(u_n-\xi_n)$ and $d_\gamma(u_n, Sv_n) \le \gamma(u_n-\mu_n)$. Substituting into (20) yields003A

$$H_\gamma(Tu_n,Sv_n) \le \max\left\{\beta_n\gamma(u_n-\xi_n), \frac{\gamma(u_n-\xi_n)+\beta_n\,\gamma(u_n-\xi_n)+\gamma(u_n-\mu_n)}{2}, \frac{\gamma(u_n-\mu_n)+(1-\beta_n)\gamma(u_n-\xi_n)}{2}\right\}$$
$$\le \max\left\{\beta_n, \frac{1-\beta_n}{2}\right\}\gamma(u_n-\xi_n)+\frac{1}{2}\gamma(u_n-\mu_n)$$

Since $\lim_{n\to\infty} sup\beta_n < 1$ then we can choose n_0 large enough to make $\max\left\{\beta_n, \frac{1-\beta_n}{2}\right\} < 1$ and condition (14) is satisfied from (19), we obtain

$$H_\gamma(Tu_n, Sp) \le \max\left\{\gamma(u_n-p), \frac{d_\gamma(u_n,Tu_n)+d_\gamma(P,SP)}{2}, \frac{d_\gamma(u_n,Sp)+d_\gamma(p,Tu_n)}{2}\right\}$$
$$\le \gamma(u_n-p)+\frac{d_\gamma(u_n,Tu_n)}{2}+\frac{d_\gamma(p,Tu_n)}{2} +\frac{1}{2}\max\{d_\gamma(P, SP), d_\gamma(u_n,Sp)\}$$

So, $\alpha=1$, $\mu=\delta=\beta=\frac{1}{2}$, and condition (15) is satisfied

Finally, from (19), we get

$$H_\gamma(Tp,Sp) \le \max\left\{\gamma(p-p), \frac{d_\gamma(p,Tp)+d_\gamma(p, Sp)}{2}, \frac{d_\gamma(p,Sp)+d_\gamma(p,Tp)}{2}\right\}$$
$$=\left\{\frac{d_\gamma(p,Tp)}{2}+\frac{d_\gamma(p, Sp)}{2}\right\}$$
$$=\frac{1}{2}\{d_\gamma(p,Tp)+d_\gamma(p, Sp)\}$$

and the condition (16) is satisfied.

Corollary (3.4)

Letμ_γ be a*CCRMS*, $\varnothing \ne A \subset \mu_\gamma$be convex $S,T: A \to CB(A)$ satisfies

$$H_\gamma(Tu,Sv) \le qmax\left\{\gamma(u-v), \frac{d_\gamma(v,Sv)[1+d_\gamma(u,Tu)]}{1+\gamma(u-v)}, \frac{d_\gamma(u,Sv)[1+d_\gamma(u,Tu)+d_\gamma(v,Tu)]}{2[1+\gamma(u-v)]}\right\} \quad (21)$$

$\exists\ u,v\ in\ A$ where $0<q<1$. If there exists an u_0 in $A \ni \{u_n\}$ satisfying (5) and (13) for$\alpha_n+\beta_n=1$

$\liminf_{n\to\infty}\alpha_n > 0$, $\lim_{n\to\infty}\beta_n = 0$, converges to p, there p is a fixed point of T.

Proof: Using (21), we get

$$H_\gamma(Tu_n, Sv_n) \le q\max\left\{\gamma(u_n - v_n), \frac{d_\gamma(v_n, Sv_n)[1 + d_\gamma(u_n, Tu_n)]}{1 + \gamma(u_n - v_n)}, \frac{d_\gamma(u_n, Sv_n)[1 + d_\gamma(u_n, Tu_n) + d_\gamma(v_n, Tu_n)]}{2[1 + \gamma(u_n - v_n)]}\right\}$$
$$\le q\max\left\{\gamma(u_n - v_n), \frac{\gamma(u_n - \mu_n)[1 + \gamma(u_n - \xi_n)]}{1 + \gamma(u_n - \xi_n)}, \frac{\gamma(u_n - \mu_n)[1 + \gamma(u_n - \xi_n) + \gamma(v_n - \xi_n)]}{2[1 + \gamma(u_n - \xi_n)]}\right\}$$

From condition (5),

$$v_n - \mu_n = (1-\beta_n)u_n + \beta_n\xi_n - \beta_n\mu_n - (1-\beta_n)\mu_n$$

So,

$$\gamma(v_n - \mu_n) \le (1-\beta_n)\gamma(u_n - \mu_n) + \beta_n\gamma(\xi_n - \mu_n)$$
$$\le (1-\beta_n)\gamma(u_n - \mu_n) + \beta_n[\gamma(u_n - \mu_n) + \gamma(u_n - \xi_n)]$$
$$= \gamma(u_n - \mu_n) + \beta_n\gamma(u_n - \xi_n)$$

Also, from condition (5) $\gamma(u_{n+1} - u_n) = \alpha_n\gamma(u_n - \mu_n)$. Since $\{u_n\}$ is convergent, $\lim_{n\to\infty}\gamma(u_{n+1} - u_n) = 0$ and from $\liminf_{n\to\infty}\alpha_n > 0$. Yields $\lim_{n\to\infty}\gamma(u_n - \mu_n) = 0$. Therefore, for all n sufficiently large,

$$\gamma(u_n - \mu_n) + \beta_n\gamma(u_n - \xi_n) \le 1 + \beta_n\gamma(u_n - \xi_n).$$

Thus, for all n sufficiently large and from inequalities in the proof of corollary (13), we have

$$\frac{\gamma(v_n - \mu_n)[1 + \gamma(u_n - \xi_n)]}{1 + \beta_n\gamma(u_n - \xi_n)} \le \gamma(v_n - \mu_n) + \frac{\gamma(v_n - \mu_n)\gamma(u_n - \xi_n)}{1 + \beta_n\gamma(u_n - \xi_n)} \le \gamma(v_n - \mu_n) + \gamma(u_n - \xi_n)$$
$$\le \gamma(u_n - \mu_n) + \beta_n\gamma(u_n - \xi_n) + \gamma(u_n - \xi_n) = (1 + \beta_n)\gamma(u_n - \mu_n) + \gamma(u_n - \mu_n)$$

Since,

$$v_n - \xi_n = (1-\beta_n)u_n + \beta_n\xi_n - \beta_n\xi_n - (1-\beta_n)\xi_n$$

So,

$$\gamma(v_n - \xi_n) = (1-\beta_n)\gamma(u_n - \xi_n)$$

Since

$$\frac{\gamma(u_n - \mu_n)[1 + \gamma(u_n - \xi_n) + \gamma(v_n - \xi_n)]}{2[1 + \beta_n\gamma(u_n - \xi_n)]} = \frac{\gamma(u_n - \mu_n)[1 + (2 - \beta_n)\gamma(u_n - \xi_n)]}{2[1 + \beta_n\gamma(u_n - \xi_n)]}$$
$$\le \frac{1}{2}[\gamma(u_n - \mu_n) + (2 - \beta_n)\gamma(u_n - \xi_n)]$$

Then for all n sufficiently large, we get:

$$H_\gamma(Tu_n, Sv_n) \le q\max\{\beta_n\gamma(1+\beta_n)\gamma(u_n - \xi_n) + \gamma(u_n - \mu_n), \frac{1}{2}[\gamma(u_n - \mu_n) + (2 - \beta_n)\gamma(u_n - \xi_n)]$$
$$\le \max\{q\beta_n, q(1+\beta_n), q(2-\beta_n)/2\}\gamma(u_n - \xi_n) + q\gamma(u_n - \mu_n)$$

and (14) is satisfied, since $\lim_{n\to\infty}\beta_n = 0$. Again from (21), we get

Since (14) is satisfied

$$H_\gamma(Tu_n,Sp)\le qmax\left\{\gamma(u_n-p),\frac{d_\gamma(p,Sp)[1+\gamma(u_n-\xi_n)]}{1+\gamma(u_n-p)},\frac{d_\gamma(u_n,Sp)[1+\gamma(u_n-\xi_n)+\gamma(p-\xi_n)]}{2[1+\gamma(u_n-p)}\right\}$$
$$\le q\gamma(u_n-p)+qmax\left\{\frac{1+\gamma(u_n-\xi_n)}{1+\gamma(u_n-p)},\frac{1+\gamma(u_n-\xi_n)+\gamma(p-\xi_n)}{2[1+\gamma(u_n-p)]}\right\}$$

$$\begin{aligned}\gamma(u_n-\xi_n)&\le\gamma(u_n-\mu_n)+\gamma(\mu_n-\xi_n)\\&\le\gamma(u_n-\mu_n)+H_\gamma(Tu_n,Sv_n)+\epsilon_n\le\gamma(u_n-\mu_n)\\&\quad+\alpha\gamma(u_n-\mu_n)+\beta\gamma(u_n-\xi_n)+\epsilon_n\end{aligned}$$

where $\alpha,\beta>0$ and $\beta<1$ Since $\lim_{n\to\infty}\gamma(u_n-\mu_n)=0$, we have $\lim_{n\to\infty} sup\ \gamma(u_n-\xi_n)\le\beta\ \lim_{n\to\infty}\sup\gamma(u_n-\xi_n)$, *since* $0\le\beta<1$, which implies that $\lim_{n\to\infty}\gamma(u_n-\xi_n)=0$ since $\gamma(p-\xi_n)\le\gamma(p-u_n)+\gamma(u_n-\xi_n)$ it follows that

$$\lim_{n\to\infty}\max\left\{\frac{1+\gamma(u_n-\xi_n)}{1+\gamma(u_n-p)},\frac{1+\gamma(u_n-\xi_n)+\gamma(p-\xi_n)}{2[1+\gamma(u_n-p)]}\right\}=\max\left\{1,\frac{1}{2}\right\}=1$$

Therefore, for all n sufficiently large (15) is satisfied. Since (14) and (15) are satisfied, then by theorem (13) $p$is a fixed point of S from (21), we obtain

$$H_\gamma(Sp,Tp)\le qmax\left\{0,d_\gamma(p,Sp)[1+d_\gamma(p,Tp)],\frac{1}{2}d_\gamma(p,Sp)[1+d_\gamma(p,Tp)+d_\gamma(p,Tp)]\right\}=0$$

and (16) is satisfied trivially.

References

[1] H. Nakano, Modular Semi-ordered Spaces, Tokyo Mathematical Book Series, Maruzen Co. Ltd, Tokyo, Japan, 1950.
[2] J. Musielak, W. Orlicz, On modular spaces, Stud. Math. 18 (1959) 49−65.
[3] V. Chistyakov, Modular metric spaces I Basic concepts, Nonlinear Anal. 72 (2010) 1−14.
[4] V. Chistyakov, Modular metric spaces II Application to superposition operators, Nonlinear Anal. 72 (2010) 15−30.
[5] D. Turkoglu, N. Manav, Fixed point theorems in a new type of modular metric spaces, Fixed Point Theory Appl. 25 (2018) 1−10.
[6] S.S. Abed, M.F. Abdul Jabbar, Equivalence between iterative schemes in modular spaces, J. Interdiscipl. Math. 22 (2019) 1529−1535.
[7] H.K. Nashine, W.I. Rabha, Symmetric solutions of nonlinear fractional integral equations via a new fixed point theorem under FG-contractive condition, Numer. Funct. Anal. Optim. 40 (2019) 1−19.
[8] M. Bachar, Nonlinear Fredholm equations in modular function spaces, Electron. J. Differ. Equ. 36 (2019) 1−9.
[9] M.A. Khamsi, A convexity property in modular function spaces, Math. Jpn. 44 (1996) 269−279.
[10] S.H. Khan, M. Abbas, Approximating fixed points of multivalued p-nonexpansive mappings in modular function spaces, Fixed Point Theory Appl. 34 (2014) 1687−1812.
[11] S.H. Khan, M. Abbas, S. Ali, Fixed point approximation of multivalued p-quasi-nonexpansive mappings in modular function spaces, J. Nonlinear Sci. Appl. 10 (2017) 3168−3179.
[12] M. Abbas, B.E. Rhoades, Fixed point theorems for two new classes of multivalued mappings, Appl. Math. Lett. 22 (2009) 1364−1368.
[13] S.S. Abed, On invariant best approximation in modular spaces, Global J. Pure Appl. Math. 13 (2017) 5227−5233.
[14] S.S. Abed, A.K.E. Sada, Common fixed points in modular spaces, Ibn Al-Haitham J. Pure Appl Sci. (2017). IHSCICONF (Special Issue).
[15] S.S. Abed, M.F.Abdul Jabbar , Convergence of iteration scheme to fixed point in modular spaces, Iraqi J. Sci. 60 (2019) 2196−2201.
[16] S.S. Abed, R.F. Abbas, S-iteration for general quasi multivalued contraction mappings, IJAMSS 5 (2016) 9−22.