Karbala International Journal of Modern Science

Volume 6 | Issue 4 Article 3

The Revised NIM for Solving the Non-Linear System Variant Boussinesq Equations and Comparison with NIM

Oday Ahmed Jasim
University of Mosul, Mosul, odayalnoamy@uomosul.edu.iq

Recommended Citation
Jasim, Oday Ahmed (2020) "The Revised NIM for Solving the Non-Linear System Variant Boussinesq Equations and Comparison with NIM," *Karbala International Journal of Modern Science*: Vol. 6 : Iss. 4 , Article 3.
Available at: https://doi.org/10.33640/2405-609X.1829

Abstract

This research aims to guide researchers to use a new method, and it is the Revised New Iterative Method (RNIM) to solve partial differential equation systems and apply them to solve problems in various disciplines such as chemistry, physics, engineering and medicine. In this paper, the numerical solutions of the nonlinear Variable Boussinesq Equation System (VBE) were obtained using a new modified iterative method (RNIM); this was planned by (Bhaleker and Datterder-Gejj). A numerical solution to the Variable Boussinesq Equation System (VBE) was also found using a widely known method, a new iterative method (NIM). By comparing the numerical solutions of RNIM and NIM methods, the solution VBE using the new revised iterative process (RNIM), is accurate, reliable, robust, promising and quickly arrives at the exact solution. Moreover, the results also show that the solution of the (RNIM) is more reliable and faster compared to the new iterative method. Finally, a solution by RNIM is easy to calculate, but sometimes requires more calculations.

Keywords

Revised New Iterative Method (RNIM), Variant Boussinesq Equation (VBE), New Iterative Method (NIM), Partial Differential Equations (PDE), Numerical Solution.

Cover Page Footnote

“The authors are very grateful to University of Mosul / College of Basic Education for their provided facilities, which helped to improve the quality of this work”.

1. Introduction

Researchers around the world seek to find and improve numerical methods that help them solve equations, especially regular and partial differential equations, partial and ordinary differential equations which have entered into all sciences and have been used to represent most phenomena in various sciences.

In this paper, the RNIM was used to solve the system (Eq. (1)) numerically through the nonlinear variant Boussinesq equation (VBE) [1,2]:

$$\begin{aligned} &E_t + Q_x + EE_x = 0, \\ &Q_t + (EQ)_x + E_{xxx} = 0. \end{aligned} \tag{1}$$

It can be written as:

$$\begin{aligned} \frac{\partial E}{\partial t} &= -\frac{\partial Q}{\partial x} - E\frac{\partial E}{\partial x} \\ \frac{\partial Q}{\partial t} &= -E\frac{\partial Q}{\partial x} - Q\frac{\partial E}{\partial x} - \frac{\partial^3 E}{\partial x^3} \end{aligned} \tag{2}$$

With an exact solution for a system (Eq. (2)) [1] is:

$$\begin{aligned} E(x,t) &= -1 + \cosh\left(\frac{x+t}{2}\right) \\ Q(x,t) &= -1 + \cosh\left(\frac{x+t}{2}\right) - \frac{1}{2}\left[-1 + \cosh\left(\frac{x+t}{2}\right)\right]^2 \end{aligned} \tag{3}$$

Some researchers use the VBE system; solving the system of nonlinear modified Boussinesq equations and find the numerical solution modified NIM for the system variant Boussinesq equation [3,4]. The variant Boussinesq equation plays an important role in fluid dynamics. Various wave solutions of (Eq. (1)) have studied numerical or analytical methods such as; homogeneous balance method, exp-function, homotopy analysis, extended Fan's sub-equation method, bifurcation theory, and algebraic methods. In this research, we find numerical solution by using RNIM and NIM methods for the system variant Boussinesq equation [4].

RNIM is easy to implement using a computer program to obtain a better result. The revised new iterative method was planned by (Bhaleker and Datterder-Gejj) to solve a nonlinear functional equations system and find convergence for the proposed method [5]. “The problem of finding a model describing biological species living together” was solved by using the RNIM [6].

The NIM was planned by (Jafari and Daftardar-Gejji), and proved by Hemeda [7,8]. The benefit of NIM is that it gives a highly accurate solution with a comparatively much lesser number of iterations. “Bhalekar and Daftardar-Gejji Apply NIM to PDE and submit a research article about convergence of NIM [9,10]. Many researchers used the NIM method in order to solve: fractional boundary value problems with Dirichlet boundary conditions using NIM”, linear and nonlinear Klein−Gordon equations, and the system of linear differential equations [11−13]. Using NIM, “the approximate analytical solutions of the Newell−Whitehead−Segel equation and solution of fractional gas dynamics and coupled Burger's equations” were solved [14,15] in addition to the solutions of Eigen value problems and Variational problems [16,17]. The NIM also contributed to “solving nonlinear Burger's equation and coupled Burger's equations, as well as solving modified Korteweg-de Vries (MKdV) System from three equations. “There was a comparison between NIM and natural homotopy perturbation method for solving nonlinear Time-fractional Wave-Like equations with variable coefficients nonlinear dynamics and systems theory” [18−20].

The arrangement for this paper is as follows: In section 1, a historical brief survey and introduction for RNIM and NIM are provided. Section 2 explains mathematical methods for the RNIM and NIM. The numerical solution of the nonlinear variant Boussinesq equation (VBE) system is found by using two methods discussed in section 3. Section 4 shows results and discourses of which the solution for RNIM is more accurate and faster compared to other commonly used technologies such as NIM. Concluding remarks are given in section 5.

2. Mathematical model

2.1. New iterative method (NIM)

In the following public functional equation:

$$E = f + M(E), \tag{4}$$

N is a “nonlinear operator”. The solution of (Eq. (2)) has the series form:

$$E = \sum_{i=0}^{\infty} E_i, \tag{5}$$

Where M is analyzed as:

https://doi.org/10.33640/2405-609X.1829

$$M\Big(\sum_{i=0}^{\infty}E_i\Big)=M(E_0)+\sum_{i=1}^{\infty}\Big[M\Big(\sum_{j=0}^{i}E_j\Big)-M\Big(\sum_{j=0}^{i-1}E_j\Big)\Big] \quad (6)$$

Equations (Eq. (4)) and (Eq. (5)) above were used in (Eq. (3)):

$$\sum_{i=0}^{\infty}E_i=f+M(E_0)+\sum_{i=1}^{\infty}\Big[M\Big(\sum_{j=0}^{i}E_j\Big)-M\Big(\sum_{j=0}^{i-1}E_j\Big)\Big] \quad (7)$$

We determine the relationship of repetition as:

$$\begin{aligned}&E_1=M(E_0)\\&E_2=M(E_0+E_1)-M(E_0)\\&E_3=M(E_o+E_1+E_2)-M(E_o+E_1)\\&E_{n+1}=M(E_o+E_1+\ldots+E_n)=M(E_o+E_1+\ldots+E_{n-1});\\&\qquad n=1,2,\ldots\ldots\end{aligned} \quad (8)$$

Then as fallowing

$$E_0+E_1+\ldots+E_{n+1}=M(E_0+E_1+\ldots+E_n);$$

$$n=1\,,2\,,\ldots\ldots. \quad (9)$$

And

$$\sum_{i=0}^{\infty}E_i=f+M\Big(\sum_{j=0}^{\infty}E_j\Big)$$

The n-term “approximate solution” of (Eq. (3)) was given as:

$$E=E_1+E_2+\ldots+E_{n-1}. \quad (10)$$

To know the convergence of NIM, see the source [10].

2.2. *Revised new iterative method (RNIM)*

We submit the “algorithm of the RNIM” provided by (Bhalekar and Daftardar-Gejji) to explain the technique [9]. We consider using (Eq. (4)) as the following:

Initial conditions (n = 0):

$$E_{i,0}=f_i,\quad i=1,2,3,\ldots\ldots,n \quad (11)$$

The first iteration (n = 1):

$$\begin{aligned}E_{1,1}&=M_1(E_{1,o},E_{2,o},E_{3,o},\ldots\ldots\ldots\ldots,E_{n,o})\\E_{2,1}&=M_2(E_{1,o}+E_{1,1},E_{2,o},E_{3,o},\ldots\ldots\ldots\ldots,E_{n,o})\\E_{3,1}&=M_3(E_{1,o}+E_{1,1},E_{2,o}+E_{2,1},E_{3,o},\ldots\ldots\ldots\ldots,E_{n,o})\\&\vdots\\E_{n,1}&=M_n(E_{1,o}+E_{1,1},E_{2,o}+E_{2,1},\ldots\ldots\ldots,\ldots\ldots\ldots\ldots,E_{n-1,0}+E_{n-1,1},E_{n,0});\quad n=1,2,3,\ldots.\end{aligned}$$

k-th iteration (n = h = 2, 3, 4,…...)

$$\begin{aligned}E_{1,h}&=M_1\Big(\sum_{i=o}^{h-1}E_{1,i},\ldots\ldots,\sum_{i=o}^{h-1}E_{n,i}\Big)-M_1\Big(\sum_{i=o}^{h-2}E_{1,i},\ldots\ldots,\sum_{i=o}^{h-2}E_{n,i}\Big)\\E_{2,k}&=M_2\Big(\sum_{i=o}^{h}E_{1,i},\sum_{i=o}^{h-1}E_{2,i},\ldots\ldots,\sum_{i=o}^{h-1}E_{n,i}\Big)-M_2\Big(\sum_{i=o}^{h-1}E_{1,i},\sum_{i=o}^{h-2}E_{2,i},\ldots\ldots,\sum_{i=o}^{h-2}E_{n,i}\Big)\\E_{j,k}&=M_j\Big(\sum_{i=o}^{h}E_{1,i},\ldots\ldots,\sum_{i=o}^{h-1}E_{j-1,i},\sum_{i=o}^{h-1}E_{j,i},\ldots\ldots,\sum_{i=o}^{h-1}E_{n,i}\Big)-M_j\Big(\sum_{i=o}^{h-1}E_{1,i}\ldots\ldots\sum_{i=o}^{h-1}E_{j-1,i},\sum_{i=o}^{h-2}E_{j,i}\ldots\ldots\sum_{i=o}^{h-1}E_{n,i}\Big)\\&\vdots\\E_{n,h}&=M_n\Big(\sum_{i=0}^{h}E_{1,i}\ldots\ldots\sum_{i=0}^{h}E_{n-1,i},\sum_{i=0}^{h-1}E_{n,i}\Big)-M_n\Big(\sum_{i=0}^{h-1}E_{1,i}\ldots\ldots\sum_{i=0}^{h-1}E_{n-1,i},\sum_{i=0}^{h-2}E_{n,i}\Big)\end{aligned}$$

Thus

$$M_i(E_i)=M_i\left(\sum_{j=0}^{\infty}E_{1,j}\ldots\ldots\sum_{j=0}^{\infty}E_{n,j}\right)=\sum_{j=1}^{\infty}E_{i,j}$$

Hence,

$$E_i=\sum_{j=1}^{\infty}E_{i,j} \tag{12}$$

3. Numerical results

In this section, we find the numerical solution for E(x, t), F(x, t) satisfying the VBE (Eq. (1)) using two methods [1,2]:

3.1. New iterative method (NIM)

To obtain the initial conditions from (Eq. (3)), let t = 0 then:

$$E(x,0)=E_0=-1+\coth\left(\frac{1}{2}x\right)$$
$$Q(x,0)=Q_0=1-\coth\left(\frac{1}{2}x\right)-\frac{1}{2}\left(-1+\coth\left(\frac{1}{2}x\right)\right)^2 \tag{13}$$

We construct two iterative E, Q for the system VBE (Eq. (2)). To solve the system of that equation by NIM and from the equation (Eq. (9)) as follows:

$$E(x,t)=\int_0^t M_1(E,Q)dt=\int_0^t\left(-\frac{\partial Q}{\partial x}-E\frac{\partial E}{\partial x}\right)dt$$
$$Q(x,t)=\int_0^t M_2(E,Q)dt=\int_0^t\left(-E\frac{\partial Q}{\partial x}-Q\frac{\partial E}{\partial x}-\frac{\partial^3 E}{\partial x^3}\right)dt \tag{14}$$

Substitution of the initial conditions (Eq. (13)) in (Eq. (4)), get:

$$E_1(x,t)=\int_0^t M_1(E_0,Q_0)dt=\int_0^t\left(-\frac{\partial Q_0}{\partial x}-E_0\frac{\partial E_0}{\partial x}\right)dt$$

$$Q_1(x,t)=\int_0^t M_2(E_0,Q_0)dt=\int_0^t\left(-E_0\frac{\partial Q_0}{\partial x}-Q_0\frac{\partial E_0}{\partial x}-\frac{\partial^3 E_0}{\partial x^3}\right)dt$$

An integral equation, we get:

$$E_1(x,t)=-\frac{t(\cosh(\chi)+1)^2)}{\sinh(\chi)^2}$$
$$Q_1(x,t)=\frac{t(+1+\cosh(\chi)^2+2\cosh(\chi))}{\sinh(\chi)^3} \tag{15}$$

Then, find E_2:

$$E_2(x,t)=\int_0^t(M_1(E_0+E_1,Q_0+Q_1,)-M_1(E_0,Q_0))dt$$

$$E_2(x,t)=\int_0^t\left(\left(-\frac{\partial}{\partial x}(Q_0+Q_1)-(E_0+E_1)\frac{\partial}{\partial\chi}(E_0+E_1)\right)-\left(-\frac{\partial}{\partial\chi}Q_0-E_0\frac{\partial}{\partial\chi}E_0\right)\right)dt \tag{16}$$

By integral equation (16), the result of E_2 will be as follows:

$$E_2(\chi,t)=\frac{1}{6}\frac{1}{\sinh(\chi)^5}\left((3\cosh(\chi)^4+2\cosh(\chi)^3t+6\cosh(\chi)^3+6\cosh(\chi)^2t+6\cosh(\chi)-6\cosh(\chi)+2t-3)t^2\right)$$

To find Q_2

$$Q_2(x,t)=\int_0^t(M_2(E_0+E_1,Q_0+Q_1)-M(E_0,Q_0))dt$$

$$Q_2(x,t)=\int_0^t\left(\left(-(E_0+E_1)\frac{\partial}{\partial\chi}(Q_0+Q_1)-(Q_0+Q_1)\frac{\partial}{\partial\chi}(E_0+E_1)-\frac{\partial^3}{\partial\chi^3}(E_0+E_1)\right)-\left(-E_0\frac{\partial}{\partial\chi}Q_0-Q_0\frac{\partial}{\partial\chi}E_0-\frac{\partial^3}{\partial\chi^3}E_0\right)\right)dt \tag{17}$$

An integral equation (Eq. (17)), get the following:

$$Q_2(x,t)=\frac{-1}{6}\left(t^2\left((3\cosh(\chi)^2+4\cosh(\chi)t+3\cosh(\chi)+6t-6\right)/\left(\cosh(\chi)^3-3\cosh(\chi)^2+3\cosh(\chi)-1\right)\right)$$

Now find E_3:

$$E_3(x,t)=\int_0^t (M_1(E_0+E_1+E_2,Q_0+Q_1+Q_2,)-M_1(E_0+E_1,Q_0+Q_1))dt$$

$$E_3(x,t)=\int_0^t \left(\left(-\frac{\partial}{\partial\chi}(Q_0+Q_1+Q_2)-(E_0+E_1+E_2)\frac{\partial}{\partial\chi}(E_0+E_1+E_2)\right)\right.$$
$$\left.-\left(-\frac{\partial}{\partial\chi}(Q_0+Q_1)-(E_0+E_1)\frac{\partial}{\partial\chi}(E_0+E_1)\right)\right)dt \tag{18}$$

Integrally (Eq. (18)), the result will be:

$$E_3(\chi,t)=\frac{-1}{2520}*\frac{1}{(\sinh(\chi)^3(\cosh(\chi)^4-4\cosh\chi^3+6\cosh(\chi)^2-4\cosh(\chi)+1))}$$
$$(t^3(-840+1050\cosh(\chi)^4\sinh(\chi)t+\ldots\ldots\ldots\ldots+\ldots\ldots\ldots\ldots+1470\mathrm{t}+504\cosh\,(\chi)\sinh(\chi)\mathrm{t}^2))$$

Now, find E, then:

$$E(x,t)=\sum_{i=0}^{n+1}E_i(x,t)=E_0(x,t)+E_1(x,t)+\ \ldots\ldots+E_{n+1}(x,t)$$

$$E(\chi,t)=E_0+E_1+E_2+E_3=\frac{1}{2520}*\frac{1}{(\sinh(\chi)^3\ (\cosh(\mathrm{x})^4-4\cosh(\mathrm{x})^3+6\cosh(\mathrm{x})^2-4\cosh(\mathrm{x})+1))}(-2520$$
$$+7560\cosh(\mathrm{x})^4\sinh(\mathrm{x})t-\ldots\ldots\ldots\ldots+\ldots\ldots\ldots+2520\sinh(\mathrm{x})^7) \tag{19}$$

To find Q_3 :

$$Q_3(x,t)=\int_0^t (M_2(E_0+E_1+E_2,Q_0+Q_1+Q_2)-M_2(E_0+E_{11},Q_0+Q_1))dt$$

$$Q_3(x,t)=\int_0^t \left(\left(-(E_0+E_1+E_2)\frac{\partial}{\partial\chi}(Q_0+Q_1+Q_2)-(Q_0+Q_1+Q_2)\ \frac{\partial}{\partial\chi}(E_o+E_1+E_2)\ -\frac{\partial^3}{\partial\chi^3}(E_o+E_1+E_2)\right)\right.$$
$$\left.-\left(-(E_o+E_1)\frac{\partial}{\partial\chi}(Q_0+Q_1)-(Q_0+Q_1)\frac{\partial}{\partial\chi}(E_0+E_1)-\frac{\partial^3}{\partial\chi^3}(\mathrm{E}_0+\mathrm{E}_1)\right)\right)dt \tag{20}$$

By integral equation (Eq. (20)), get:

$$Q_3(x,t) = \frac{1}{2520} * \frac{1}{\cosh(x)^6 - 6\sinh(x)^5 + 15\cosh(x)^4 - 20\cosh(x)^3 + 15\cosh(x)^2 - 6\cosh(x) + 1}$$
$$\left(t^3\left(-2520 + 2100\cosh(x)^3\sinh(x)t + \ldots\ldots\ldots + \ldots\ldots\ldots + \ldots\ldots\ldots + 6300t + 1680\cosh(x)\sinh(x)t^2\right)\right) \tag{21}$$

Now to find $Q(x,t)$, then:

$$Q(x,t) = \sum_{i=0}^{n+1} Q_i(x,t) = Q_0(x,t) + Q_1(x,t) + \ldots + Q_{n+1}(x,t)$$

$$Q(x,t) = Q_0 + Q_1 + Q_2 + Q_3 = \frac{-1}{2520} * \frac{1}{\left(\cosh(x)^6 - 6\cosh(x)^5 + 15\cosh(x)^4 - 20\cosh(x)^3 + 15\cosh(x)^2 - 6\cosh(x) + 1\right)}$$
$$\left(-2520 - 2520\cosh(x)^4\sinh(x)t - 10080\cosh(x)^3\sinh(x)t + \ldots\ldots\ldots + \ldots\ldots\ldots + 9540\cosh(x)\cosh(x)t^4 - 420\cosh(x)^4\sinh(x)t^3\right) \tag{22}$$

The numerical solution and behavior gave the absolute errors and mean square error between the exact solution and NIM for E, Q and various values of t, and $x = 5$ for the variant Boussinesq equations as shown in Table 1 and Fig. 1 respectively.

3.2. Revised new iterative method

Obtain the initial conditions from (3) and let t = 0, then:

$$E(x,0) = E_0 = -1 + \coth\left(\frac{1}{2}x\right)$$
$$Q(x,0) = Q_0 = 1 - \coth\left(\frac{1}{2}x\right) - \frac{1}{2}\left(-1 + \coth\left(\frac{1}{2}x\right)\right)^2 \tag{23}$$

We construct two iterative E, F for the system VBE (Eq. (2)). To solve the system of that equation by RNIM and from the equation (Eq. (9)), get:

$$E(x,t) = \int_0^t M_1(E,Q)dt = \int_0^t \left(-\frac{\partial Q}{\partial x} - E\frac{\partial E}{\partial x}\right)dt$$
$$Q(x,t) = \int_0^t M_2(E,Q)dt = \int_0^t \left(-E\frac{\partial Q}{\partial x} - Q\frac{\partial E}{\partial x} - \frac{\partial^3 E}{\partial x^3}\right)dt \tag{24}$$

Table 1
The absolute errors and mean square error between "NIM and exact solution" for E, Q and diverse values of t and $x = 5$ for the variant Boussinesq equations.

t	$\lvert E_{exact} - E_{NIM}\rvert$	$\lvert Q_{exact} - Q_{NIM}\rvert$
0.01	$3.554762343*10^{-10}$	$2.2 *10^{-10}$
0.02	$9.143072000*10^{-9}$	$1.8*10^{-10}$
0.03	$1.960219500*10^{-9}$	$6.8*10^{-10}$
0.04	$2.517161815*10^{-9}$	$2.72*10^{-9}$
0.05	$5.731206700*10^{-9}$	$5.75*10^{-9}$
0.06	$1.213672629*10^{-8}$	$1.228*10^{-8}$
0.07	$1.938946680*10^{-8}$	$2.196*10^{-8}$
0.08	$3.192572529*10^{-8}$	$3.745*10^{-8}$
0.09	$4.912779520*10^{-8}$	$5.949 *10^{-8}$
0.1	$7.388454817*10^{-8}$	$9.09 *10^{-8}$
MSE	$8.972096230*10^{-16}$	$1.385255583 *10^{-15}$

Substitution of the initial conditions (Eq. (23)) in (Eq. (24)), get:

$$E_1(x,t)=\int_0^t M_1(E_0,Q_0)dt=\int_0^t\left(-\frac{\partial F_0}{\partial x}-E_0\frac{\partial E_0}{\partial x}\right)dt$$
$$Q_1(x,t)=\int_0^t M_2(E_0,Q_0)dt=\int_0^t\left(-(E_0+E_1)\frac{\partial}{\partial \chi}Q_0-Q_0\frac{\partial}{\partial \chi}(E_0+E_1)-\frac{\partial^3}{\partial \chi^3}(E_0+E_1)\right)dt \tag{25}$$

Through integration (Eq. (25)), get:

$$E_1(x,t)=-\frac{t(\cosh(x)+1)}{\sinh(x)^2}$$

$$Q_1(x,t)=-\frac{1}{2}\frac{1}{\sinh(x)^5}\left((t\cosh(\chi)^4-2\cosh(\chi)^4+6t\cosh(\chi)^3-4\cosh(\chi)^3+12t\cosh(\chi)^2+10t\cosh(\chi)+10\cosh(\chi)+3t+2)t\right)$$

To find E_2:

$$E_2(x,t)=\int_0^t(M_1(E_0+E_1,Q_0+Q_1,)-M_1(E_0,Q_0))dt$$

$$E_2(x,t)=\int_0^t\left(\left(-\frac{\partial}{\partial \chi}(Q_o+Q_1)-(E_o+E_1)\frac{\partial}{\partial \chi}(E_o+E_1)\right)-\left(-\frac{\partial}{\partial \chi}Q_o-E_0\frac{\partial}{\partial \chi}E_o\right)\right)dt \tag{26}$$

And by integral equation (Eq. (26)), the result is as follows:

$$E_2(x,t)=\frac{-1}{6}\left(t^2\left(t\cosh(\chi)^2-3\cosh(x)\sin(\chi)+9t\cosh(\chi)-2t\sinh(\chi)+3\sinh(\chi)+10t\right)/\left(\cosh(\chi)^3-3\cosh(\chi)^2+3\cosh(\chi)-1\right)\right) \tag{27}$$

To Find Q_2

$$Q_2(x,t)=\int_0^t(M_2(E_0+E_1+E_2,V_0+V_1)-M(U_0,V_0)\times)dt$$

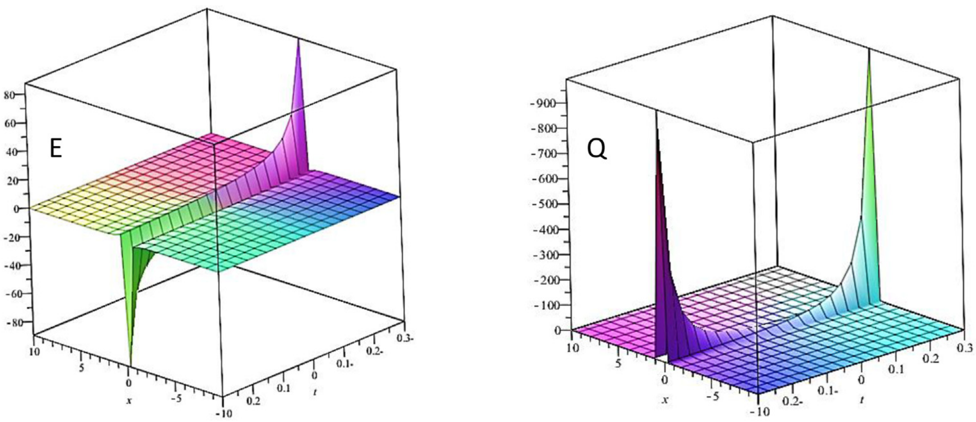

Fig. 1. “The exact solution of the system VBE” (Eq. (3)) when $x=-10......10$, $t=-0.3......0.3$.

$$Q_2(x,t)=\int_0^t\left(\left(-(E_0+E_1+E_2)\frac{\partial}{\partial\chi}(Q_0+Q_1)-(Q_0+Q_1)\frac{\partial}{\partial\chi}(E_0+E_1+E_2)-\frac{\partial^3}{\partial\chi^3}(E_o+E_1+E_2)\right)-\left(-(E_o+E_1)\frac{\partial}{\partial\chi}Q_o-Q_0\frac{\partial}{\partial\chi}(E_o+E_1)-\frac{\partial^3}{\partial x^3}(E_o+E_1)\right)\right)dt \quad (28)$$

By integral equation (Eq. (28)), get:

$$Q_2(x,t)==\frac{1}{360}*\frac{1}{(\cosh(\chi)^6-6\sinh(\chi)^5+\ldots\ldots+\ldots\ldots+1)}\big(t^2\big(-360+1020\,t-15\cosh(\chi)^4\sinh(\chi)t^2-36\cosh(\chi)^3\sinh(\chi)t^3-\ldots\ldots+\ldots\ldots+1440\cosh(x)^2+1260\cosh(\chi)-540\sinh(\chi)\,t^2\big)\big) \quad (29)$$

Now to find $E_{3,}$ then:

$$E_3(x,t)=\int_0^t(M_1(E_0+E_1+E_2,Q_0+Q_1+Q_2)-M_1(E_0+E_1,Q_0+Q_1))dt$$

$$E_3(x,t)=\int_0^t\left(\left(-\frac{\partial}{\partial\chi}(Q_0+Q_1+Q_2)-(E_o+E_1+E_2)\frac{\partial}{\partial\chi}(E_o+E_1+E_2)\right)-\left(-\frac{\partial}{\partial\chi}(Q_o+Q_1)-(E_o+E_1)\frac{\partial}{\partial\chi}(E_o+E_1)\right)\right)dt \quad (30)$$

Integrally (Eq. (30)), get:

$$E_3(x,t) = \frac{-1}{2520} * \frac{1}{(\cosh(x)^7 - 7\cosh(x)^6 + \ldots\ldots + \ldots\ldots + \ldots\ldots + \ldots\ldots + 7\cosh(x) - 1)} \left(t^3\left(-5040 - 10080\,t + 3549\cosh(x)^5 t^2 + \ldots\ldots + \ldots\ldots - 16800\cosh(x)^2 + 840\sinh(x)t - 315\cosh(x)^5\sinh(x)t\right)\right) \quad (31)$$

To find E, then:

$$E(x,t) = \sum_{i=0}^{n+1} E_i(x,t) = E_0(x,t) + E_1(x,t) + \ldots\ldots + E_{n+1}(x,t)$$

$$E(x,t) = E_0 + E_1 + E_2 + E_3 = \frac{-1}{2520} * \frac{1}{(\cosh(\chi)^7 - 7\cosh(\chi)^6 + \ldots\ldots + \ldots\ldots + \ldots\ldots + \ldots\ldots + 7\cosh(\chi) - 1)} \left(t^3\left(-2520 + 2520t + 5985\cosh(\chi)^3\sinh(\chi)t^4 + \ldots\ldots + \ldots\ldots - 17283\sinh(\chi)t^5 - 12270\sinh(\chi)^5 t^7\right)\right) \quad (32)$$

To find Q_3 :

$$Q_3(x,t) = \int_0^t \left(M_2(E_0 + E_1 + E_2 + E_3, Q_0 + Q_1 + Q_2) - M_2(E_0 + E_1 + E_2, Q_0 + Q_1)\right)dt$$

$$Q_3(x,t) = \int_0^t \left(\left(-(E_0 + E_1 + E_2 + E_3)\frac{\partial}{\partial\chi}(Q_0 + Q_1 + Q_2) - (Q_0 + Q_1 + Q_2)\frac{\partial}{\partial\chi}(E_o + E_1 + E_2 + E_3) - \frac{\partial^3}{\partial\chi^3}(E_o + E_1 + E_2 + E_3)\right) - \left(-(E_o + E_1 + E_2)\frac{\partial}{\partial\chi}(Q_0 + Q_1) - (Q_0 + Q_1)\frac{\partial}{\partial\chi}(E_0 + E_1 + E_2) - \frac{\partial^3}{\partial\chi^3}(E_0 + E_1 + E_2)\right)\right)dt \quad (33)$$

By integral equation (Eq. (33)), get:

$$Q_3(x,t) = \frac{1}{3632428800} * \frac{1}{(\cosh(x)^{13} - 13\sinh(x)^{12} + 78\cosh(x)^{11} - \ldots\ldots\ldots\ldots\ldots\ldots + \ldots\ldots\ldots\ldots\ldots + 13\cosh(x) + 1)}$$
$$(t^3(- 10291881600 - 2875672800t + \ldots\ldots\ldots\ldots + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$+30421591200\cosh(x)^{10}t - 299497980600\cosh(x)^{12}t)) \quad (34)$$

Now to find $Q(x,t)$, then:

$$Q(\chi,t) = \sum_{i=0}^{n+1} Q_i(\chi,t) = Q_0(\chi,t) + Q_1(\chi,t) + \ldots + Q_{n+1}(\chi,t)$$
$$Q(\chi,t) = Q_0 + Q_1 + Q_2 + Q_3$$
$$= \frac{1}{3632428800} * \frac{1}{(\cosh(x)^{13} - 13\sinh(x)^{12} + 78\cosh(x)^{11} - \ldots\ldots\ldots\ldots\ldots\ldots - 78\cosh(x)^2 + 13\cosh(x) - 1)}(t^3($$
$$- 3632428800 - 1078831353600\cosh(x)^5 t^2 + - \ldots\ldots\ldots\ldots\ldots\ldots\ldots + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + \ldots\ldots\ldots\ldots\ldots$$
$$+ 100610509272(\cosh(x)^7 t^{11} - 120472582230\cosh(x)^9 t^8)) \quad (35)$$

The numerical solution and behavior gave the absolute errors and mean square error between RNIM and exact solution for E, Q and various values of t and $x = 5$ for the variant Boussinesq equations as shown in Table 2 and Fig. 2 respectively.

The numerical solutions obtained by the RNIM and NIM methods are shown for diverse values of t and x in Tables 3 and 4 and Fig. 3 respectively in addition to the comparison of absolute errors and mean square error for the variant Boussinesq equation system between the RNIM and NIM for E, Q was conducted.

Table 2
The absolute errors and mean square error between “RNIM and exact solution” for E, Q and diverse values of t and $x = 5$ for the variant Boussinesq equations.

t	$\lvert E_{exact} - E_{RNIM}\rvert$	$\lvert Q_{exact} - Q_{RMIN}\rvert$
0.1	$1.290089022*10^{-9}$	$2.275482100*10^{-10}$
0.2	$1.050088123*10^{-9}$	$2.897600000*10^{-11}$
0.3	$1.370000000*10^{-9}$	$5.977280000*10^{-11}$
0.4	$1.340000000*10^{-9}$	$3.554886000*10^{-10}$
0.5	$4.400000000*10^{-10}$	$2.282700000*10^{-12}$
0.6	$2.000000000*10^{-11}$	$4.019130000*10^{-10}$
0.7	$2.090000000*10^{-9}$	$4.446300000*10^{-11}$
0.8	$7.900827483*10^{-10}$	$6.987170000*10^{-11}$
0.9	$5.900000000*10^{-10}$	$5.768891000*10^{-10}$
1	$1.330000000*10^{-9}$	$5.654739000*10^{-10}$
MSE	$1.135054524*10^{-18}$	$5.8936714410*10^{-20}$

4. Results and discussions

a. By comparing Fig. 1 (the exact solution of the system VBE” (Eq (3)) when $x = -10\ldots\ldots10$, $t = -0.3\ldots\ldots0.3$) with Fig. 2 (The E and Q by NIM, for various values of $x = -10\ldots.10$, $t = -0.3\ldots0.3.$) and Fig. 3 (The E and Q by RNIM, for various values of $x = -10\ldots.10$, $t = -0.3\ldots0.3$), it is evident that Fig. 1 is very similar to Fig. 3, which means that the solution by method RMIN is very near to exact solution.
b. From the comparison of Tables 1 and 2, the absolute error ($E_{exact} - E_{RNIM}$, $Q_{exact} - Q_{RNIM}$, $E_{exact} - E_{NIM}$, $Q_{exact} - Q_{NIM}$) for RNIM and NIM for all values of t and $x = 5$ for the nonlinear system (VBE). We note that the clear superiority of the RMIN method is because the error amount of the RNIM method is less than the error amount of the NIM method for E, Q respectively. This means that the RNIM method is the best because it is

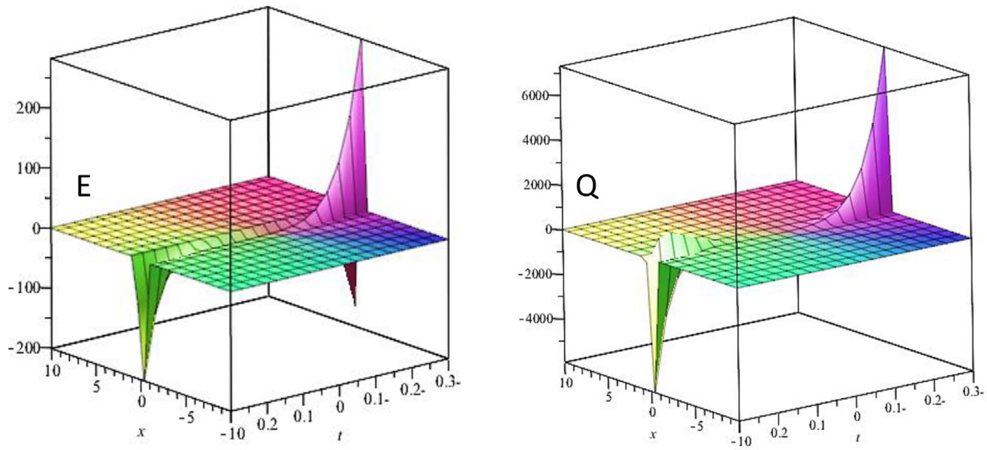

Fig. 2. The E and Q by NIM, for various values of $x = -10 \ldots 10, t = -0.3 \ldots 0.3$.

Table 3
Comparison of "absolute errors and mean square error" for E by the variant Boussinesq equations system between RNIM with NIM for varied values of x and t.

(t,x)	E_{exact}	E_{RNIM}	E_{NIM}	$\lvert E_{exact} - E_{RNIM}\rvert$	$\lvert E_{exact} - E_{NIM}\rvert$
(10.1,0.05)	0.000078155	0.000078155	0.00007815501747	0	$2.7020*10^{-10}$
(10.2,0.06)	0.000067268	0.000067268	0.00006726845486	0	$5.8037*10^{-10}$
(10.3,0.07)	0.000057898	0.000057899	0.00005789614112	$1*10^{-9}$	$1.85888*10^{-9}$
(10.4,0.08)	0.000049833	0.000049833	0.00004982925760	0	$3.74240*10^{-9}$
(10.5,0.09)	0.000042892	0.000042891	0.00004288335697	$1*10^{-9}$	$8.79673*10^{-9}$
(10.6,0.1)	0.000036917	0.000036917	0.00003690213064	0	$1.486936*10^{-8}$
(10.7,0.11)	0.000031775	0.000031774	0.00003174903207	$1*10^{-9}$	$2.596793*10^{-8}$
(10.8,0.12)	0.000027349	0.000027349	0.00002730809547	0	$4.090453*10^{-8}$
(10.9,0.13)	0.000023539	0.000023540	0.00002348164333	$1*10_{-9}$	$5.735667*10^{-8}$
(11,0.14)	0.000020260	0.000020259	0.00002018144877	$1*10^{-9}$	$7.808715*10^{-8}$
MSE				$1.505619958*10^{-19}$	$1.221576748*10^{-15}$

closer to the exact solution and the absolute errors are less for it.

c. In Tables 1 and 2, the mean square error (MSE) by NIM for E, Q is $8.972096230*10^{-16}$ and $1.385255583*10^{-15}$ respectively while the mean square error for RNIM for E, Q is $1.135054524*10^{-18}$ and $5.8936714410*10^{-20}$

Table 4
Comparison of "absolute errors and mean square error" for Q by the variant Boussinesq equations system between RNIM with NIM for varied values of x and t.

(t,x)	Q_{exact}	Q_{RNIM}	Q_{NIM}	$\lvert Q_{exact} - Q_{RNIM}\rvert$	$\lvert Q_{exact} - Q_{NIM}\rvert$
(10.1,0.05)	−0.00007815805410	−0.00007815827154	−0.00007815825021	6.88924×10^{-10}	8.245541×10^{-11}
(10.2,0.06)	−0.00006727026249	−0.00006727071534	−0.00006727041069	4.31077×10^{-10}	3.951478×10^{-10}
(10.3,0.07)	−0.00005789967609	−0.00005789989884	−0.00005789851728	4.40×10^{-12}	7.703609×10^{-8}
(10.4,0.08)	−0.00004983424166	−0.00004983450320	−0.00004983059171	1.560×10^{-10}	9.523873×10^{-6}
(10.5,0.09)	−0.00004289291986	−0.00004289265787	−0.00004288410207	1.3995×10^{-10}	9.338900×10^{-5}
(10.6,0.1)	−0.00003691768143	−0.00003691783508	−0.00003690193847	3.5842×10^{-10}	1.851572×10^{-5}
(10.7,0.11)	−0.00003177550483	−0.00003177531804	−0.00003174892675	2.9246×10^{-10}	2.828288×10^{-6}
(10.8,0.12)	−0.00002734937398	−0.00002734916087	−0.00002730881075	6.0465×10^{-10}	9.258313×10^{-8}
(10.9,0.13)	−0.00002353927704	−0.00002353957285	−0.00002348164087	4.0137×10^{-10}	2.848026×10^{-9}
(11,0.14)	−0.00002026020523	−0.00002026066741	−0.00002018151581	1.8008×10^{-10}	1.449418×10^{-11}
MSE				$6.968375906 \times 10^{-22}$	$1.227095381 \times 10^{-15}$

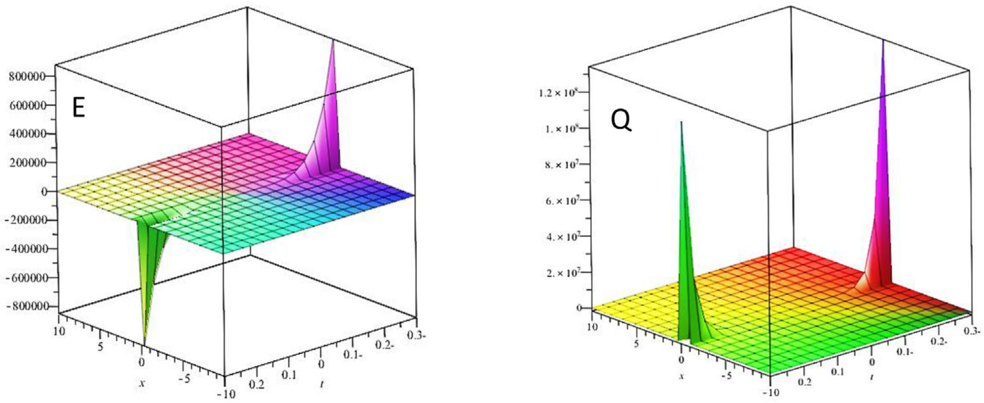

Fig. 3. The E and Q by RNIM, for various values of $x = -10\ldots10, t = -0.3\ldots0.3$

respectively, then the mean square error (MSE) for RNIM is less which means that RNIM is better.

d. From Tables 3 and 4 comparison of "absolute errors and mean square error" for E, Q by the variant Boussinesq equations system between RNIM with NIM for varied values of x and t. RNIM is more accurate and faster compared to the NIM method.

e. Fig. 4: during the drawing of E, Q For values of $x = 11$, $t = 0\ldots2$, it appears that the figure drawn with the RNIM method is closer to the actual shape compared to the figure drawn with the NIM method.

f. All tables and figures are obtained using Maple 18.

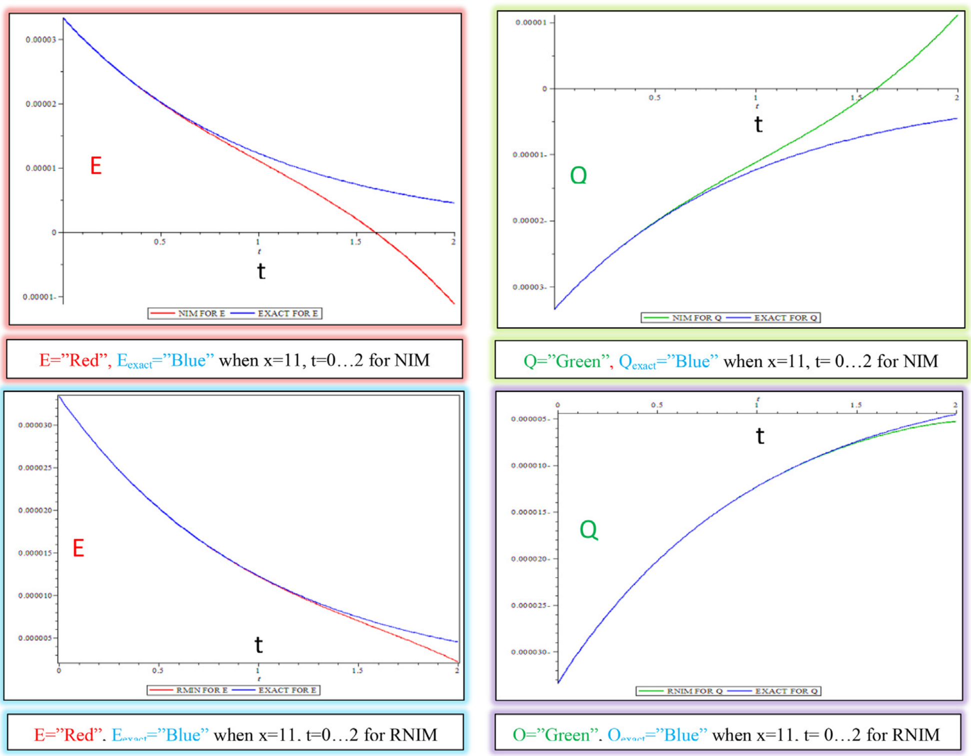

Fig. 4. The approximation E, Q by to methods RNIM and NIM with the exact solution, for varied values of $x = 11$, $t = 0\ldots2$

5. Conclusion

The Revised NIM (RNIM) has been successfully applied for obtaining an excellent approximation chain of variant Boussinesq equations. It has been shown that the Revised New Iterative Method (RNIM) is quite capable and suited for finding exact solutions; this is because the RNIM technique every time you perform an optimization depends on the previous step of the same equation. This is clear from the explanation of the method in Section (2.2). The consistency of the method gives this method wider applicability. By comparing the numerical solution of the VBE system and using the RNIM and NIM methods with the exact solution, it was found that the RNIM method is the most accurate and fastest way to reach the exact solution. Maple 18 was used to carry out the computations. Finally, it is worth noting that this method is clear and concise, and can be applied to non-linear PDE equations in engineering and applied science.

Acknowledgements

"The authors are very grateful to University of Mosul/College of Basic Education for their provided facilities, which helped to improve the quality of this work".

References

[1] A.H.A. Ali, A.A. Soliman, New exact solutions of some nonlinear partial differential equations modified extended tanh-function method applications MKdV equation, Int. J. Nonlinear Sci. 5 (2008) 79−88.
[2] R.L. Sachs, On the Integrable variant of the Boussinesq system: painlevé property, rational solutions, a related many-body system, and equivalence with the AKNS hierarchy, Phys. Nonlinear Phenom. 30 (1988) 1−27, https://doi.org/10.1016/0167-2789(88)90095-4.
[3] M. Matinfar, A. Fereidoon, A. Aliasghartoyeh, On solving nonlinear modified Boussinesq equations, Appl. Math. Sci. 3 (2009) 67−74.
[4] J.F. Lu, Modified variational iteration method for variant Boussinesq equation, Therm. Sci. 19 (2015) 1195−1199, https://doi.org/10.2298/tsci1504195l.
[5] S. Bhalekar, V. Daftardar-Gejji, Solving a system of nonlinear functional equations using revised new iterative method, Int. J. Math. Comput. Phys. Electr. Comput. Eng. 6 (2012) 8−21.
[6] A.O. Collins, O.I. Obiajulu Hassan, The revised new iterative method for solving the model describing biological species living together, Am. J. Comput. Appl. Math. 6 (2016) 136−143.
[7] V. Daftardar-Gejji, H. Jafari, An iterative method for solving nonlinear functional equations, J. Math. Anal. Appl. 316 (2006) 753−763, https://doi.org/10.1016/j.jmaa.2005.05.009.
[8] A.A. Hemeda, New Iterative Method: an Application for Solving Fractional Physical Differential Equations, Abst. Appl. Anal. 2013 (2013) 1−9, https://doi.org/10.1155/2013/617010.
[9] S. Bhalekar, V. Daftardar-Gejji, New iterative method: application to partial differential equations, Appl. Math. Comput. 203 (2008) 778−783, https://doi.org/10.1016/j.amc.2008.05.071.
[10] V. Daftardar-Gejji, S. Bhalekar, Solving fractional boundary value problems with Dirichlet boundary conditions using a new iterative method, Comput. Math. Appl. 59 (2010) 1801−1809, https://doi.org/10.1016/j.camwa.2009.08.018.
[11] S. Bhalekar, V. Daftardar-Gejji, Convergence of the new iterative method, Int. J. Diff. Equ. 2011 (2011) 1−10, https://doi.org/10.1155/2011/989065.
[12] M. Yaseen, M. Samraiz, The modified new iterative method for solving linear and nonlinear klein-gordon equations, Appl. Math. Sci. 6 (2012) 2979−2987.
[13] M.S. ur Rehman, M. Yaseen, T. Kamran, New iterative method for solution of system of linear differential equations, Int. J. Sci. Res. 5 (2016) 1287−1289.
[14] J. Patade, S. Bhalekar, Approximate analytical solutions of Newell-Whitehead-Segel equation using a new iterative method, World J. Model. Simulat. 11 (2015) 94−103.
[15] M.S. Al-luhaibi, New iterative method for fractional gas dynamics and coupled Burger's equations, Sci. World J. 2015 (2015) 1−8, https://doi.org/10.1155/2015/153124.
[16] T. Kamran, M.J. Amir, M.S. Ur Rehman, Solution of eigen value problems by using new iterative method, Int. J. Sci. Eng. Res. 6 (2015) 1355−1361.
[17] T. Kamran, M. Yaseen, M.S. Ur Rehman, Z. Ali, Solution of variational problems using new iterative method, Int. J. Sci. Eng. Res. 7 (2016) 1619−1624.
[18] Z.Y.A. Allawee, Application new iterative method for solving nonlinear Burger's equation and coupled Burger's equations, Int. J. Comput. Sci. 15 (2018) 31−35, https://doi.org/10.5281/zenodo.1292414.
[19] A.F. Qasim, B.J. Salim, Application new iterative method for solving modified Korteweg-de Vries (MKdV) system from three equations, J. Adv. Res. Dynamic Contr. Syst. 11 (2019) 1−7.
[20] A. Khalouta, A. Kadem, Comparison of new iterative method and natural homotopy perturbation method for solving nonlinear time-fractional wave-like equations with variable coefficients, Nonlinear Dynam. Syst. Theor. 19 (2019) 160−169.